MATHEMATICS - II

BCA 201

SELF LEARNING MATERIAL

DIRECTORATE

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MATHEMATICS-II

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The real number system as a complete ordered field, neighbourhood, open and closed sets, limit points of sets.

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Limits, continuity, sequential Continuity, algebra of Continuous functions, Continuity of composite functions, Continuity on (a,b) implying boundedness.

Unit – III

Sequence, convergent sequence, Cauchy Sequence, monotonic sequence, Sub-sequence, Limit superior and limit inferior of sequences.

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Unit - V

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- Ratio Analysis

- Fund flow statement.

Unit - I

The real number system as a complete ordered field

This course will deal with multivariate calculus, basically the analytical study of regions of n-dimensional real space. Before considering this, we should review what we mean by the term 'real numbers'.

The basic idea is that the real numbers are the elements of the set of real numbers. However, a set is just a set; something about which we can take subsets and do other set theoretic operations. In order to understand this, we need to consider the set of real numbers as something with more structure. The set of real numbers is actually a complete ordered field.

Let us pause to describe what this means:

Definition 1: A field is a triple (F, +, *) where F is a set and + and * are binary operators (called addition and multiplication) which satisfy the following axioms:

1. Both addition and multiplication are commutative and associative and multiplication is distributive over addition. In symbols, this means that for all a, b, and c in F, one has:

    1. a + b = b + a, a*b = b*a
    2. (a + b) + c = a + (b + c), (a*b)*c = a*(b*c)
    3. a*(b + c) = a*b + a*c

2. (Identities) There are two different elements 0 and 1 (called zero and one) in F such that for all a in F: a + 0 = a and a*1 = a.

3. (Inverses) For every a in F, there is at least one b in F such that a + b = 0. Further, if a is non-zero, there is at least one c in F such that a*b = 1. The elements b (respectively c) are called the additive (respectively the multiplicative) inverses of a and are denoted -a (respectively 1/a).

Definition 2:

An ordered field is a pair ((F, +, *), <) where (F, +, *) is a field and < is binary relation on F such that:

1. (Transitivity) For every a, b, c in F, if a < b and b < c, then a < c.

2. (Trichotomy) For every a and b in F, exactly one of the following holds:

    1. a < b
    2. a = b
    3. b < a

3. For all a, b, and c in F, if a < b, then a + c < b + c. Further, if c > 0, then a*c < b*c.

It is assumed that you have probably already had a course in which you showed that most of the standard rules of algebra follow from the assumption that one has an ordered field. (If you haven't ever done this before, then take home and work through Landau's Foundations of Analysis; he will show you in a weekend read how to derive the basic properties of the real numbers starting from Peano postulates. Just for practice, however, you might try to show:

Exercise 1 In any field, additive and multiplicative identities and inverses are unique. In an ordered field, if a < b and c < 0, then b*c < a*c.

Definition 3: A subset S of an ordered field F is bounded above by an element a in F if b < a for all b in S. The upper bound a is called a least upper bound for S if every other upper bound of S is larger than a.

Definition 4: An ordered field F is said to be complete if every subset S of F which has an upper bound, also has a least upper bound.

Exercise 2: Show that least upper bounds are unique. Mimic the above definition to define lower bounds and greatest lower bounds. Show that in complete fields, every set S with a lower bound has a greatest lower bound.

It is useful to have the notion of complete ordered field in order to sort out some basic properties of real numbers. But the real value is that these assumptions alone are enough to completely characterize the real numbers. The result is expressed in terms of:

Definition 5: An order isomorphism between two ordered fields ((F,+,*),<) and ((G,+,*),<) is a 1-1 and onto map f:F ->G such that for all a and b in f, one has:

1. f(a + b) = f(a) + f(b)
2. f(a*b) = f(a)*f(b)
3. If a < b, then f(a) < f(b).

So basically, two ordered fields are order isomorphic if we cannot distinguish between them by using their addition and multiplication operations and their order relation. In these terms the basic result is:

Theorem 1: There is a complete ordered field and any two complete ordered fields are order isomorphic.

This result will not be proved here, but is found in algebra books. Here are some of the ideas of a proof:

1. Using the elements 0 and 1, one can build up the integers through addition, and then one can take quotients to get the rational numbers Q.

2. Define a sequence $x_1, x_2, x_3, \cdots$ of elements of an ordered field to be a Cauchy sequence if for every $\epsilon > 0$, there is an n such that for any i, j greater than n, one has $|x_i - x_j| < \epsilon$. We say that the sequence $x_1, x_2, x_3, \cdots$ has limit x if for every $\epsilon > 0$, there is an n such that for every i > n, one has $|x_i - n| < \epsilon$.

3. Two sequences $x_1, x_2, x_3, \cdots$ and $y_1, y_2, y_3, \cdots$ are said to be equivalent if the sequence $x_1 - y_1, x_2 - y_2, x_3 - y_3, \cdots$ has limit 0. You can show that this is an equivalence relation.

4. If $x_1, x_2, x_3, \cdots$ and $y_1, y_2, y_3, \cdots$ are Cauchy sequences, then adding or multiplying corresponding terms gives new Cauchy sequences. Similarly, taking additive inverses gives a Cauchy sequence and so does taking multiplicative inverses if the original sequence has a non-zero limit and none of the elements are zero.

5. So that equivalence is preserved under addition and multiplication of Cauchy sequences.

6. Take the set S of Cauchy sequences with elements in the rational numbers Q. The set of equivalence classes of such sequences can be shown to be a complete ordered field.

1.2 The Cauchy-Schwartz Inequality

Henceforth, let use let R denote the real numbers, i.e. the set part of a complete ordered field. By taking n-tuples of real numbers we get n-dimensional real space, denoted R^n; its elements are called points. The points are demoted $(x^1, x^2, \ldots, x^n)$ or simply (x^i). Points can be added componentwise and we can multiply a real number by a point to get another point whose coordinates are just the original coordinates multiplied by the real number. Of course, this makes R^n into an n-dimensional vector space. But, we can also define a dot product:

$$(x^1, \ldots, x^n) \cdot (y^1, \ldots, y^n) = \sum_{i=1}^{n} x^i y^i$$

As usual, this allows us to define the length of a point as the square root of the dot product of the point with itself:

$$|(x^1, ..., x^n)| = \sqrt{\sum_{i=1}^{n} (x^i)^2}$$

Now the length function behaves like you would expect. For example:

Proposition 1 (Triangle Inequality) For any two points x and y in R^n , one has

$$|x + y| \leq |x| + |y|$$

Proof: Squaring both sides, we see this amounts to showing that

$$(x + y) \cdot (x + y) = x \cdot x + 2(x \cdot y) + y \cdot y$$

is at most

$$(|x| + |y|)^2 = |x|^2 + 2|x||y| + |y|^2$$

Comparing terms, we see that the triangle inequality is equivalent to:

Proposition 2 (Cauchy-Schwartz Inequality) For any two points x and y in R^n , one has $x \cdot y \leq |x||y|$, i.e.

$$\sum_{i=1}^{n} x^i y^i \leq \sqrt{\sum_{i=1}^{n} (x^i)^2} \sqrt{\sum_{i=1}^{n} (y^i)^2}$$

Exercise 3: Draw a picture in the case n = 2 and identify various factors of the left hand side divided by the right to convince yourself that the result is basically equivalent to the addition formula for cosines.

Recall from your calculus course, that the dot product was shown to be:

$$x \cdot y = |x||y| \cos(\theta)$$

where θ is the angle between the vectors from the origin to the points x and y respectively. This was probably only shown in the case n = 2; but it is another way of understanding why Proposition 2 should be true. In our case, we will actually do things in the opposite direction, i.e. prove that Proposition 2 is true and use it to define the angle θ .

The main use of the dot product in Calculus was to give you the means of calculating the projection of one vector on another. Using this interpretation, can you see why Proposition 2 should be true?

Given two vectors (i.e. two points) x and y, if we subtract from x the vector projection p of x on y, then we get a vector perpendicular to y. Now, geometrically, this vector should be smallest vector amongst all vectors of the form $x - \lambda y$ for all possible real λ . Geometrically, we have `solved' the problem: Minimize $|x - \lambda y|$.

Ahh! we are finally doing calculus!

So let's do calculus: To minimize this, we should minimize the square,

$$|x - \lambda y|^2 = |x|^2 - 2\lambda(x \cdot y) + |y|^2 \lambda^2$$

If x or y is zero, this is easy (What is the answer?); so assume both are non-zero. Now, if think of x and y as being constants, then this is just a parabola. So, either from your knowledge of parabolas or from elementary calculus, it is easy to see when this is minimized: Setting the derivative to zero gives $\lambda = x \cdot y / |y|^2$. Substituting back shows that the minimum value is $|x|^2 - x \cdot y / |y|^2$. Since this minimum value is non-negative, we have just proved Proposition 2.

A slight variant: We know that the expression on the right side of the displayed formula must be positive unless x is a multiple of y. But, if it is positive for all values of λ , then the roots must both be complex and so the discriminant is negative. But this is precisely the Cauchy-Schwartz inequality.

Remark: Make sure you sort out all the geometry from the proof to be sure we really have a proof here and not just heuristic hand waving. When you have done this, compare to the proof in Spivak -- hopefully, then you will understand his very slick proof.

Another approach entirely Let's go back to the notion of angle between two vectors.

Along with the dot product, you had a vector cross product in the case of R^3 . It gave you a vector that was perpendicular to the original two vectors, say v and w, and its length was precisely $|v||w||\sin\theta|$. It could also be calculated as the value of the determinant of a strange looking matrix in which the top row had the unit vectors i, j, and k in the coordinate axis directions and the other two rows had the coordinates of v and w respectively. Remember the formula:

$$(v \cdot w)^2 + |v \times w|^2 = |v|^2 |w|^2$$

(If not, then it is time to prove it -- yes, the Pythagorean Theorem in its many guises, including trigonometry, is your best friend.)

Identities are mysterious and wonderful. If you write the above in terms of all the coordinates, then you simply have an algebraic identity.

Exercise 4: Use the above identity to prove the Cauchy-Schwartz inequality. Now generalize it to n dimensions and get another proof of Proposition 2.

Properties of the Real Numbers as an Ordered Field

In this section we give 8 axioms related to the definition of the real numbers, R. All properties of sets of real numbers, limits, continuity of functions, integrals, and derivatives will follow from this definition.

Definition. A field F is a nonempty set with two operations + and · called addition and multiplication, such that:

(1) If a, b ∈ F then a + b and a · b are uniquely determined elements of F (i.e., + and · are binary operations).

(2) If a, b, c ∈ F then (a + b) + c = a + (b + c) and (a · b)· c = a ·(b · c) (i.e., + and · are associative).

(3) If a, b ∈ F then a + b = b + a and a · b = b · a (i.e., + and · are commutative).

(4) If a, b, c ∈ F then a · (b + c) = a · b + a · c (i.e., · distributes over +).

(5) There exists 0, 1 ∈ F (with 0 6= 1) such that 0 + a = a and 1 · a = a for all g ∈ F.

(6) If a ∈ F then there exists −a ∈ F such that a + (−a) = 0.

(7) If a ∈ F a 6= 0, then there exists a −1 such that a · a −1 = 1.

0 is the additive identity, 1 is the multiplicative identity, −a and a −1 are inverses of a.

Example. Some examples of fields include:

1. The rational numbers Q.

2. The rationals extended by √ 2: Q[√ 2].

3. The algebraic numbers A.

4. The real numbers R.

5. The complex numbers C.

6. The integers modulo p where p is prime Zp.

Theorem 1-3. For F a field, the additive and multiplicative identities are unique.

Theorem 1-4. For F a field and $a \in F$, the additive and multiplicative inverses of a are unique.

Theorem 1-5. For F a field, $a \cdot 0 = 0$ for all $a \in F$.

Theorem 1-6. For F a field and $a, b \in F$:

(a) $a \cdot (-b) = (-a) \cdot b = -(a \cdot b)$.

(b) $-(-a) = a$.

(c) $(-a) \cdot (-b) = a \cdot b$.

Note. We add another axiom to our development of the real numbers.

Axiom 8/Definition of Ordered Field. A field F is said to be ordered if there is $P \subset F$ (called the positive subset) such that

(i) If $a, b \in P$ then $a + b \in P$ (closure of P under addition).

(ii) If $a, b \in P$ then $a \cdot b \in P$ (closure of P under multiplication).

(iii) If $a \in F$ then exactly one of the following holds: $a \in P$, $-a \in P$, or $a = 0$ (this is The Law of Trichotomy).

Example. Q, Q[$\sqrt{2}$], A, and R is an ordered field. C and Zp are fields that are not ordered.

Definition. Let F be a field and P the positive subset. We say that $a < b$ (or $b > a$) if $b - a \in P$.

Note. The above definition allows us to compare pairs of elements of F and to "order" the elements of the field.

Exercise 1.2.5. If F is an ordered field, $a, b \in F$ with $a \leq b$ and $b \leq a$ then $a = b$.

Theorem 1-7. Let F be an ordered field. For $a, b, c \in F$:

(a) If $a < b$ then $a + c < b + c$.

(b) If $a < b$ and $b < c$ then $a < c$ ("<'' is transitive)

(c) If $a < b$ and $c > 0$ then $ac < bc$.

(d) If $a < b$ and $c < 0$ then $ac > bc$.

(e) If $a \neq 0$ then $a^2 = a \cdot a > 0$.

Note. Recall interval notation from Calculus 1 (see page 18).

Note. We have trouble defining exponentiation when the exponent is irrational (at least, for now).

Theorem 1-8. Let x be a positive real number and let n be a positive integer. Then there is a unique positive number y such that $y^n = x$.

Note. The proof of Theorem 1-8 depends on a result from the next section and we will consider it then.

Note. In Theorem 1-8, we say $y = x^{1/n} = \sqrt[n]{x}$. We define $x^{p/q} = (x^{1/q})^p$ where p and q are positive integers.

Theorem 1-9. Let x be a positive real number, and let s_1 and s_2 be positive rational numbers where $s_1 < s_2$. Then

(a) $x^{s_1} < x^{s_2}$ if $x > 1$.

(b) $x^{s_1} > x^{s_2}$ if $0 < x < 1$.

Theorem 1-10. Let x and y be positive real numbers with $x < y$ and let s be a positive rational number. Then $x^s < y^s$.

Exercise 1.2.7. Prove:

(a) $1 > 0$.

(b) If $0 < a < b$ then $0 < 1/b < 1/a$.

(c) If $0 < a < b$ then $a^n < b^n$ for natural number n.

(d) If $a > 0$, $b > 0$ and $a^n < b^n$ for some natural number n, then $a < b$.

(f) Prove Theorem 1-10.

Theorem 1-12. The Binomial Theorem.

Let a and b be real numbers and let m be a natural number. Then

$$(a + b)^m = \sum_{j=0}^{m} \binom{m}{j} a^j b^{m-j}.$$

Note. We can use Mathematical Induction to prove the Binomial Theorem (in fact, you likely did so in Math Reasoning [MATH 2800]).

Definition. For a ∈ R, the absolute value of a is

Neighbourhood

Hey there! This post talks to you about the term 'Neighborhood in mathematics' (nbd). which plays key role in every math stream. To know about neighborhood, we have to know limit& basic things such as limit point, interior point. also, here we'll discuss about its types- Circular nbd & Rectangular nbd..

We'll go step wise as further-

- Introduction
- Necessity
- Definition
- Types

Neighborhood-Introduction:

In most of the areas of mathematics, such as- Algebra,Topology,etc neighborhood is the basic & most important term. without which no function can be defined. This term resembles with the general meaning of 'Neighborhood' i.e. nearby things. Here, in mathematics, instead of things, we mention some points nearby a point.

In this definition, we can restrict the distance from an arbitrary point under consideration, so that the neighborhood must have some definiteness, about which we'll discuss further.

Necessity of Neighborhood:

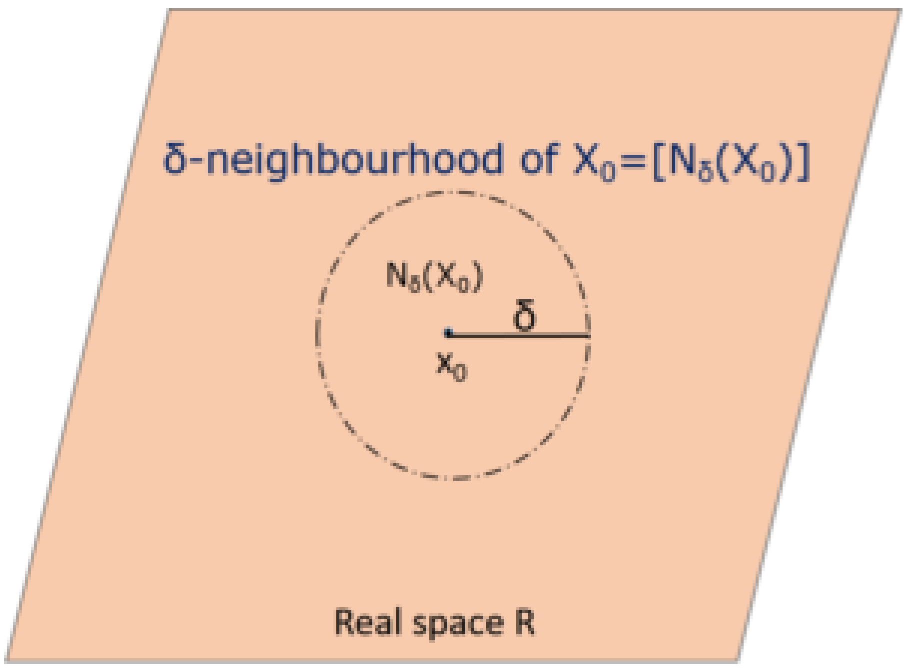
The importance of this term is highlighted in graphical representation. & the actual existence of the properties related to that arbitrary point (or set,say) under consideration can be clarified using nbd. Also, some major definitions such as-Limit,Cauchy sequence,etc make use of it.

Definition:

Let x_0 be a fixed point & δ be a distance from x_0 which is a positive real number.Then the set of all points which are at a distance δ from x_0, is the definition of Neighborhood at distance δ.i.e.δ-nbd (N_δ)

In the form of set-

$$\mathbf{N_\delta=\{x/|x-x_0|<\delta\}}$$



Definition of neighborhood

Definition in Complex plane:

Any open circle around a point z_0 at a distance R is its r-nbd. i.e.-$\{z/|z-z_0|<R\}$

Definition in Metric space:

A neighborhood of a point ‘p’ in a metric space X, is a set $N_r(p)$ i.e.r-nbd of p. We define it as-$N_r(p)=\{q/|d(p,q)<r\}$; r>0.

Where, r is the radius of the neighborhood & $N_r(p)$ is the r-nbd.

Types:

There are two main types of neighborhood:

- δ-nbd
  - Circular nbd
  - Rectangular nbd
- Deleted δ-nbd

δ-neighborhood:

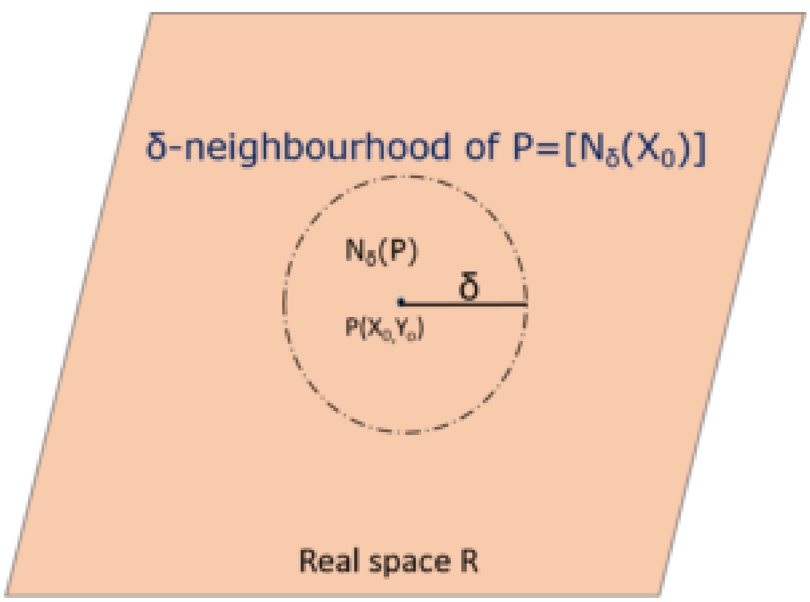
Let x_0 be a fixed point & δ be a distance from x_0 which is a positive real number.Then the set of all points which are at a distance δ from x_0, is the definition of Neighborhood at distance δ.i.e.δ-nbd (N_δ)(neighborhood).
In the form of set-

$N_\delta=\{x/|x-x_0|<0\}$

Circular neighborhood:

Set of all points P(x,y) which are at a distance δ from $P(x_0,y_0)$ so as to form shape of a circle. i.e.-

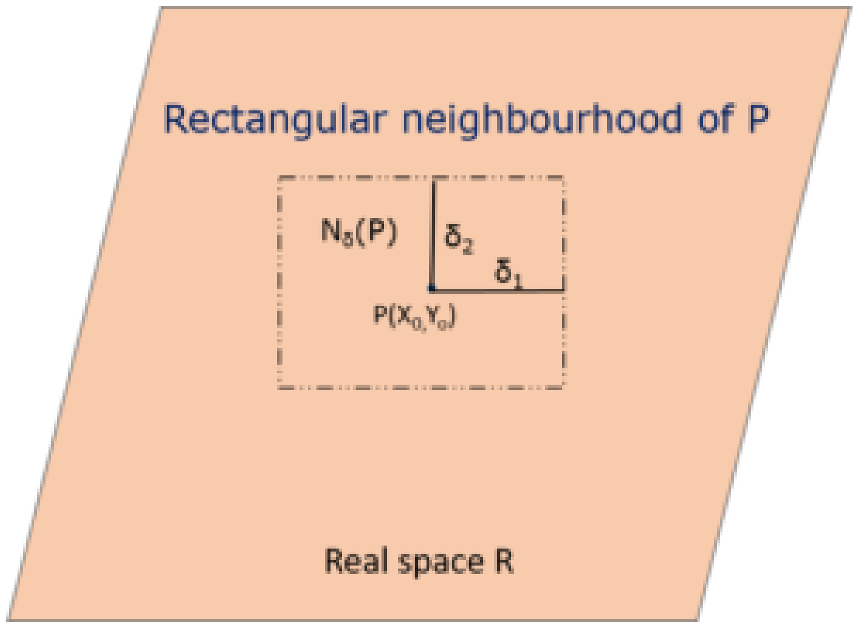
$$N_\delta=\{(x,y)\in R^2\backslash(x-x_0)^2+(y-y_0)^2<\delta\}$$



Rectangular neighborhood:

Set of all points P(x,y) within a rectangle of sides δ_1 & δ_2 from $P(x_0,y_0)$. that is-

$N_\delta=\{(x,y)\in R^2 \backslash (x-x_0)^2<\delta_1,(y-y_0)^2<\delta_2\}$



Deleted δ-neighborhood:

When $|x-x_0|>0$, also, for $x\neq x_0$, in other words, x_0 is neglected from nbd. the set so formed by combining $|x-x_0|>0$ and $|x-x_0|<\delta$. i.e.-

$0<|x-x_0|<\delta$ that means, $x_0-\delta$, $x_0+\delta$

open and closed sets

Let (X,d)(X,d) be a metric space with distance d:X×X→[0,∞)d:X×X→[0,∞).

- A point x0∈D⊂Xx0∈D⊂X is called an **interior point in D** if there is a small ball centered at x0x0 that lies entirely in DD,

x0 interior point ⇔def∃ε>0;Bε(x0)⊂D.

- A point $x_0\in X$ is called a **boundary point of D** if any small ball centered at x_0 has non-empty intersections with both D and its complement,

$$x_0 \text{ boundary point } \overset{\text{def}}{\iff} \forall \varepsilon>0\ \exists x,y\in B_\varepsilon(x_0);\ x\in D,\ y\in X\setminus D.$$

- The set of interior points in D constitutes its **interior**, $\mathrm{int}(D)$, and the set of boundary points its **boundary**, ∂D. D is said to be **open** if any point in D is an interior point and it is **closed** if its boundary ∂D is contained in D; the **closure of D** is the union of D and its boundary:

$$\overline{D}:=D\cup\partial D.$$

Alternative notations for the closue of D in X include $\overline{D}^X$, $\mathrm{clos}(D)$ and $\mathrm{clos}(D;X)$.[1)]

Ex.

- In R with the usual distance $d(x,y)=|x-y|$, the interval $(0,1)$ is open, $[0,1)$ neither open nor closed, and $[0,1]$ closed.[2)]
- The set

$$D:=\{(x,y)\in\mathbb{R}^2: x>0, y\geq 0\}$$

is neither closed nor open in Euclidean space $\mathbb{R}^2$ (metric coming from a norm, e.g., $d(x,y)=\|x-y\|_{l^2}=((x_1-y_1)^2+(x_2-y_2)^2)^{1/2}$), since its boundary contains both points $(x,0)$, $x>0$, in D and points $(0,y)$, $y\geq 0$, not in D. The closure of D is

$$\overline{D}=\{(x,y)\in\mathbb{R}^2: x\geq 0, y\geq 0\}.$$

- An entire metric space is both open and closed (its boundary is empty).
- In l^∞,

$$B_1\not\ni(1/2,2/3,3/4,\dots)\in\overline{B_1}.$$

- For a general metric space, the **closed ball**

$$\tilde{B}_r(x_0):=\{x\in X: d(x,x_0)\leq r\}$$

may be larger than the closure of a ball, Br(x0)‾ Br(x0)‾. If we let XX be a space with the discrete metric,

{d(x,x)d(x,y)=0,=1,x≠y.{d(x,x)=0,d(x,y)=1,x≠y.

Then

B1(x0)={x0}, so that B1(x0)‾={x0}‾={x0}.B1(x0)={x0}, so that B1(x0)‾={x0}‾={x0}.

But

B~1(x0)=X.

℘ (Open) balls are open

Let (X,d)(X,d) be a metric space, x0x0 a point in XX, and r>0r>0. Then Br(x0)Br(x0) is open in XX with respect to the metric dd.

Proof

Pick x∈Br(x0)x∈Br(x0). Then

d(x,x0)<r⟹∃ε>0;d(x,x0)<r−ε⟹d(y,x)<ε implies d(y,x0)≤d(y,x)+d(x,x0)<ε+(r−ε)=r.d(x,x0)<r⟹∃ε>0;d(x,x0)<r−ε⟹d(y,x)<ε implies d(y,x0)≤d(y,x)+d(x,x0)<ε+(r−ε)=r.

This means: y∈Br(x0)y∈Br(x0) if y∈Bε(x)y∈Bε(x), i.e. Bε(x)⊂Br(x0)Bε(x)⊂Br(x0).

limit points of sets

In mathematics, a limit point (or cluster point or accumulation point) of a set in a topological space is a point that can be "approximated" by points of in the sense that every neighbourhood of with respect to the topology on also contains a point of other than itself. A limit point of a set does not itself have to be an element of .

This concept profitably generalizes the notion of a limit and is the underpinning of concepts such as closed set and topological closure. Indeed, a set is closed if and only if it contains all of its limit points, and the topological closure operation can be thought of as an operation that enriches a set by uniting it with its limit points.

There is also a closely related concept for sequences. A cluster point (or accumulation point) of a sequence in a topological space is a point such that, for every neighbourhood of , there are infinitely many natural numbers such that . This concept generalizes to nets and filters.

Definition

Let be a subset of a topological space . A point in is a limit point (or cluster point or accumulation point) of if every neighbourhood of contains at least one point of different from itself.

Note that it doesn't make a difference if we restrict the condition to open neighbourhoods only. It is often convenient to use the "open neighbourhood" form of the definition to show that a point is a limit point and to use the "general neighbourhood" form of the definition to derive facts from a known limit point.

If is a space (which all metric spaces are), then is a limit point of if and only if every neighbourhood of contains infinitely many points of . In fact, spaces are characterized by this property.

If is a Fréchet–Urysohn space (which all metric spaces and first-countable spaces are), then is a limit point of if and only if there is a sequence of points in whose limit is . In fact, Fréchet–Urysohn spaces are characterized by this property.

The set of limit points of is called the derived set of .

Types of limit point

If every neighborhood of contains infinitely many points of , then is a specific type of limit point called an ω-accumulation point of .

If every neighborhood of contains uncountably many points of , then is a specific type of limit point called a condensation point of

If every neighborhood of satisfies , then is a specific type of limit point called a complete accumulation point of .

Unit - II

Limits

In the previous **section** we looked at a couple of problems and in both problems we had a function (slope in the tangent problem case and average rate of change in the rate of change problem) and we wanted to know how that function was behaving at some point x=ax=a. At this stage of the game we no longer care where the functions came from and we no longer care if we're going to see them down the road again or not. All that we need to know or worry about is that we've got these functions and we want to know something about them.

To answer the questions in the last section we choose values of xx that got closer and closer to x=ax=a and we plugged these into the function. We also made sure that we looked at values of xx that were on both the left and the right of x=ax=a. Once we did this we looked at our table of function values and saw what the function values were approaching as xx got closer and closer to x=ax=a and used this to guess the value that we were after.

This process is called **taking a limit** and we have some notation for this. The limit notation for the two problems from the last section is,

limx→12−2x2x−1=−4limt→5t3−6t2+25t−5=15limx→12−2x2x−1=−4limt→5t3−6t2+25t−5=15

In this notation we will note that we always give the function that we're working with and we also give the value of xx (or tt) that we are moving in towards.

In this section we are going to take an intuitive approach to limits and try to get a feel for what they are and what they can tell us about a function. With that goal in mind we are not going to get into how we actually compute limits yet. We will instead rely on what we did in the previous section as well as another approach to guess the value of the limits.

Both approaches that we are going to use in this section are designed to help us understand just what limits are. In general, we don't typically use the methods in this section to compute limits and in many cases can be very difficult to use to even estimate the value of a limit and/or will give the wrong value on occasion. We will look at actually computing limits in a couple of sections.

Let's first start off with the following "definition" of a limit.

Definition

We say that the limit of f(x)f(x) is LL as xx approaches aa and write this as

limx→af(x)=Llimx→a f(x)=L

provided we can make f(x)f(x) as close to LL as we want for all xx sufficiently close to aa, from both sides, without actually letting xx be aa.

This is not the exact, precise definition of a limit. If you would like to see the more precise and mathematical definition of a limit you should check out the **The Definition of a Limit** section at the end of this chapter. The definition given above is more of a "working" definition. This definition helps us to get an idea of just what limits are and what they can tell us about functions.

So just what does this definition mean? Well let's suppose that we know that the limit does in fact exist. According to our "working" definition we can then decide how close to LL that we'd like to make f(x)f(x). For sake of argument let's suppose that we want to make f(x)f(x) no more than 0.001 away from LL. This means that we want one of the following

f(x)−L<0.001if f(x) is larger than LL−f(x)<0.001if f(x) is smaller than Lf(x)−L<0.001if f(x) is larger than LL−f(x)<0.001if f(x) is smaller than L

Now according to the "working" definition this means that if we get xx sufficiently close to aa we can make one of the above true. However, it actually says a little more. It says that somewhere out there in the world is a value of xx, say XX, so that for all xx's that are closer to aa than XX then one of the above statements will be true.

This is a fairly important idea. There are many functions out there in the world that we can make as close to LL for specific values of xx that are close to aa, but there will be other values of xx closer to aa that give functions values that are nowhere near close to LL. In order for a limit to exist once we get f(x)f(x) as close to LL as we want for some xx then it will need to stay in that close to LL (or get closer) for all values of xx that are closer to aa. We'll see an **example** of this later in this section.

In somewhat simpler terms the definition says that as xx gets closer and closer to x=ax=a (from both sides of course…) then f(x)f(x) **must** be getting closer and closer to LL. Or, as we move in towards x=ax=a then f(x)f(x) **must** be moving in towards LL.

It is important to note once again that we must look at values of xx that are on both sides of x=ax=a. We should also note that we are not allowed to use x=ax=a in the definition. We will often use the information that limits give us to get some information about what is going on right at x=ax=a, but the limit itself is not concerned with what is actually going on at x=ax=a. The limit is only concerned with what is going on around the point $x=ax=a$. This is an important concept about limits that we need to keep in mind.

An alternative notation that we will occasionally use in denoting limits is

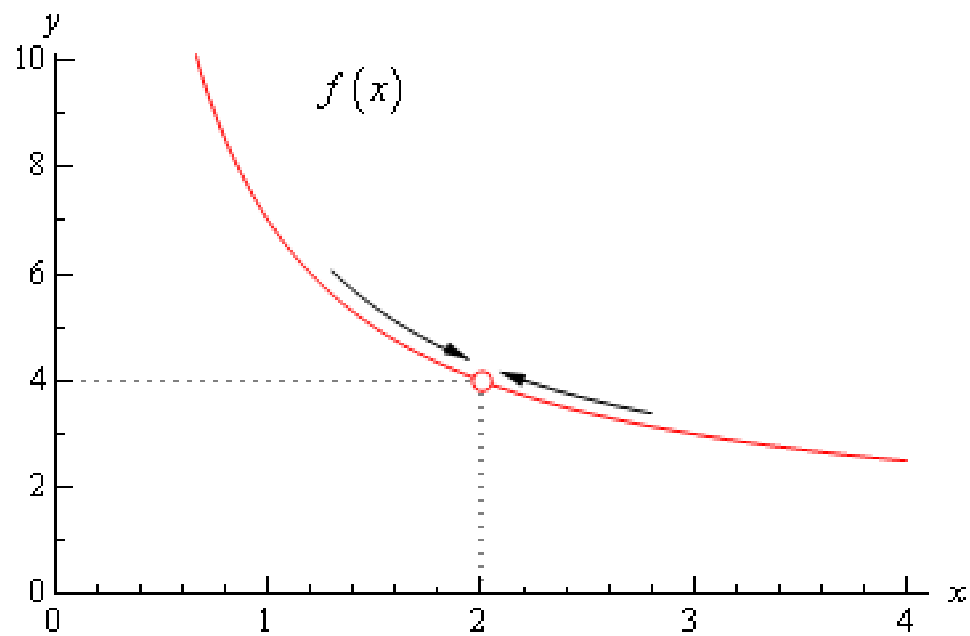
$f(x) \to L as x \to a f(x) \to L as x \to a$

How do we use this definition to help us estimate limits? We do exactly what we did in the previous **section**. We take xx's on both sides of $x=ax=a$ that move in closer and closer to aa and we plug these into our function. We then look to see if we can determine what number the function values are moving in towards and use this as our estimate.

Let's work an example.

Example 1 Estimate the value of the following limit.$\lim x \to 2 x2+4x-12x2-2x \lim x \to 2 x2+4x-12x2-2x$
Show Solution

Let's think a little bit more about what's going on here. Let's graph the function from the last example. The graph of the function in the range of xx's that were interested in is shown below.



First, notice that there is a rather large open dot at $x=2x=2$. This is there to remind us that the function (and hence the graph) doesn't exist at $x=2x=2$.

As we were plugging in values of xx into the function we are in effect moving along the graph in towards the point as $x=2x=2$. This is shown in the graph by the two arrows on the graph that are moving in towards the point.

When we are computing limits the question that we are really asking is what yy value is our graph approaching as we move in towards $x=ax=a$ on our graph. We are **NOT** asking what yy value the graph takes at the point in question. In other words, we are asking what the graph is doing **around** the point $x=ax=a$. In our case we can see that as xx moves in towards 2 (from both sides) the function is approaching $y=4y=4$ even though the function itself doesn't even exist at $x=2x=2$. Therefore, we can say that the limit is in fact 4.

So, what have we learned about limits? Limits are asking what the function is doing **around** $x=ax=a$ and are **not** concerned with what the function is actually doing at $x=ax=a$. This is a good thing as many of the functions that we'll be looking at won't even exist at $x=ax=a$ as we saw in our last example.

Let's work another example to drive this point home.

Example 2 Estimate the value of the following limit.limx→2g(x)where,g(x)= ⎧⎨⎩ x2+4x−12x2−2xif x≠26if x=2limx→2g(x)where,g(x)={x2+4x−12x2−2xif x≠26if x=2

Show Solution

Let's make the point one more time just to make sure we've got it. Limits are **not** concerned with what is going on at $x=ax=a$. Limits are only concerned with what is going on **around** $x=ax=a$. We keep saying this, but it is a very important concept about limits that we must always keep in mind. So, we will take every opportunity to remind ourselves of this idea.

Since limits aren't concerned with what is actually happening at $x=ax=a$ we will, on occasion, see situations like the previous example where the limit at a point and the function value at a point are different. This won't always happen of course. There are times where the function value and the limit at a point are the same and we will eventually see some examples of those. It is important however, to not get excited about things when the function and the limit do not take the same value at a point. It happens sometimes so we will need to be able to deal with those cases when they arise.

Let's take a look another example to try and beat this idea into the ground.

Example 3 Estimate the value of the following limit.limθ→01−cos(θ)θlimθ→01−cos(θ)θ

Show Solution

So, once again, the limit had a value even though the function didn't exist at the point we were interested in.

It's now time to work a couple of more examples that will lead us into the next idea about limits that we're going to want to discuss.

Example 4 Estimate the value of the following limit.$\lim_{t \to 0} \cos(\pi t)$$\lim_{t \to 0} \cos(\pi t)$
Show Solution

This last example points out the drawback of just picking values of the variable and using a table of function values to estimate the value of a limit. The values of the variable that we chose in the previous example were valid and in fact were probably values that many would have picked. In fact, they were exactly the same values we used in the problem before this one and they worked in that problem!

When using a table of values there will always be the possibility that we aren't choosing the correct values and that we will guess incorrectly for our limit. This is something that we should always keep in mind when doing this to guess the value of limits. In fact, this is such a problem that after this section we will never use a table of values to guess the value of a limit again.

This last example also has shown us that limits do not have to exist. To that point we've only seen limits that existed, but that just doesn't always have to be the case.

Let's take a look at one more example in this section.

Example 5 Estimate the value of the following
limit.$\lim_{t \to 0} H(t)$where,$H(t)=\{0 \text{ if } t<0 \; 1 \text{ if } t\geq 0$$\lim_{t \to 0} H(t)$where,$H(t)=\{0 \text{ if } t<0 \; 1 \text{ if } t\geq 0$
Show Solution

Let's summarize what we (hopefully) learned in this section. In the first three examples we saw that limits do not care what the function is actually doing at the point in question. They only are concerned with what is happening around the point. In fact, we can have limits at $x=a$$x=a$ even if the function itself does not exist at that point. Likewise, even if a function exists at a point there is no reason (at this point) to think that the limit will have

the same value as the function at that point. Sometimes the limit and the function will have the same value at a point and other times they won't have the same value.

Next, in the third and fourth examples we saw the main reason for not using a table of values to guess the value of a limit. In those examples we used exactly the same set of values, however they only worked in one of the examples. Using tables of values to guess the value of limits is simply not a good way to get the value of a limit. This is the only section in which we will do this. Tables of values should always be your last choice in finding values of limits.

The last two examples showed us that not all limits will in fact exist. We should not get locked into the idea that limits will always exist. In most calculus courses we work with limits that almost always exist and so it's easy to start thinking that limits always exist. Limits don't always exist and so don't get into the habit of assuming that they will.

Finally, we saw in the fourth example that the only way to deal with the limit was to graph the function. Sometimes this is the only way, however this example also illustrated the drawback of using graphs. In order to use a graph to guess the value of the limit you need to be able to actually sketch the graph. For many functions this is not that easy to do.

There is another drawback in using graphs. Even if you have the graph it's only going to be useful if the yy value is approaching an integer. If the yy value is approaching say −15123−15123 there is no way that you're going to be able to guess that value from the graph and we are usually going to want exact values for our limits.

So, while graphs of functions can, on occasion, make your life easier in guessing values of limits they are again probably not the best way to get values of limits. They are only going to be useful if you can get your hands on it and the value of the limit is a "nice" number.

The natural question then is why did we even talk about using tables and/or graphs to estimate limits if they aren't the best way. There were a couple of reasons.

First, they can help us get a better understanding of what limits are and what they can tell us. If we don't do at least a couple of limits in this way we might not get all that good of an idea on just what limits are.

The second reason for doing limits in this way is to point out their drawback so that we aren't tempted to use them all the time!

We will eventually talk about how we really do limits. However, there is one more topic that we need to discuss before doing that. Since this section has already gone on for a while we will talk about this in the next section.

Continuity

Continuity, in mathematics, rigorous formulation of the intuitive concept of a function that varies with no abrupt breaks or jumps. A function is a relationship in which every value of an independent variable—say *x*—is associated with a value of a dependent variable—say *y*. Continuity of a function is sometimes expressed by saying that if the *x*-values are close together, then the *y*-values of the function will also be close. But if the question "How close?" is asked, difficulties arise.

For close *x*-values, the distance between the *y*-values can be large even if the function has no sudden jumps. For example, if $y = 1{,}000x$, then two values of *x* that differ by 0.01 will have corresponding *y*-values differing by 10. On the other hand, for any point *x*, points can be selected close enough to it so that the *y*-values of this function will be as close as desired, simply by choosing the *x*-values to be closer than 0.001 times the desired closeness of the *y*-values. Thus, continuity is defined precisely by saying that a function $f(x)$ is continuous at a point x_0 of its domain if and only if, for any degree of closeness ε desired for the *y*-values, there is a distance δ for the *x*-values (in the above example equal to 0.001ε) such that for any *x* of the domain within the distance δ from x_0, $f(x)$ will be within the distance ε from $f(x_0)$. In contrast, the function that equals 0 for *x* less than or equal to 1 and that equals 2 for *x* larger than 1 is not continuous at the point $x = 1$, because the difference between the value of the function at 1 and at any point ever so slightly greater than 1 is never less than 2.

A function is said to be continuous if and only if it is continuous at every point of its domain. A function is said to be continuous on an interval, or subset of its domain, if and only if it is continuous at each point of the interval. The sum, difference, and product of continuous functions with the same domain are also continuous, as is the quotient, except at points at which the denominator is zero. Continuity can also be defined in terms of limits by saying that $f(x)$ is continuous at x_0 of its domain if and only if, for values of *x* in its domain,

A more abstract definition of continuity can be given in terms of sets, as is done in topology, by saying that for any open set of *y*-values, the corresponding set of *x*-values is also open. (A set is “open” if each of its elements has a “neighbourhood,” or region enclosing it, that lies entirely within the set.) Continuous functions are the most basic and widely studied class of functions in mathematical analysis, as well as the most commonly occurring ones in physical situations.

sequential Continuity

Theorem 1. Let S ⊂ R n . Let a ∈ S and let f : S → R m. Then f is continuous at a if and only if f(xn) → f(a) for all sequences xn ∈ S, xn → a.

Proof. First suppose f is continuous at a. Let xn ∈ S, xn → a. Let > 0 be given. Chose δ > 0 so that if ||x − a|| < δ then ||f(x) − f(a)|| < . Now choose N so that if n > N then ||xn − a|| < δ. Then ||f(xn) − f(a)|| < δ, so f(xn) → f(a). Notice this is correct even when a is an isolated point of S.

Next suppose f is not continuous at a. If f is not continuous at a then a cannot be an isolated point, since every function is continuous at an isolated point of its domain. If f is not continuous there is some for which no matter how what δ we choose there is a point xn ∈ S with ||f(xn) − f(a)|| ≥ . So let’s take δ = 1/n and xn ∈ S, ||xn − a|| < 1/n, ||f(xn)

− f(a)|| ≥ . Then xn → a but f(xn) 6→ f(a). Hence some sequence of points xn converges to a but f(xn) does not converge to f(a)

Notice the equivalence does not require proof at at isolated point, since every function is continuous at an isolated point and every sequence xn that converges to an isolated point satisfies xn = a for large enough n.

algebra of Continuous functions

If f and g are continuous at x_0 then

(1) f + g is continuous at $x = x_0$,

(2) f - g is continuous at $x = x_0$,

(3)f . g is continuous at $x = x_0$, and

(4) f/g is continuous at $x = x_0$ $(g(x) \neq 0)$.

(5) Composite function theorem on continuity.

If f is continuous at $g(x_0)$ and g is continuous at x_0 then fog is continuous at x_0.

Continuity in a closed interval

Definition 9.9

A function f :[a , b] → R is said to be continuous on the closed interval [a, b] if it is continuous on the open interval (a, b) and

$\lim_{x \to a} f(x) = f(a)$ and $\lim_{x \to b} f(x) = f(b)$.

That is, the function f is continuous from the right at a and continuous from the left at b, and is continuous at each point $x_0 \in (a, b)$.

Illustration 9.7

Discuss the continuity of $f(x) = \sqrt{[1-x^2]}$

The domain of definition of f is the closed interval [-1,1].

(f is defined if $1 - x^2 \geq 0$)

For any point c∈ (-1,1)

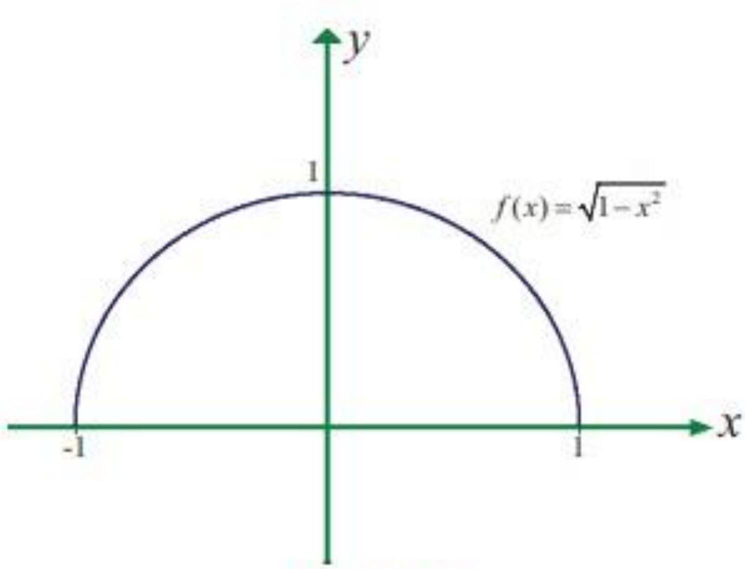


Fig. 9.36

$$\lim_{x \to c} f(x) = \lim_{x \to c} \sqrt{1-x^2} = \left[\lim_{x \to c}(1-x^2)\right]^{\frac{1}{2}}$$

$$= (1-c^2)^{\frac{1}{2}} = f(c).$$

$$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} (1-x^2)^{\frac{1}{2}} = 0 = f(1).$$

$$\lim_{x \to -1^-} f(x) = \left[\lim_{x \to -1^-} (1-x^2)\right]^{\frac{1}{2}} = 0 = f(-1).$$

Thus f is continuous on [-1, 1]. One can also solve this problem using composite function theorem.

Example 9.37

Describe the interval(s) on which each function is continuous.

(i) $f(x) = \tan x$

(ii) $g(x) = \begin{cases} \sin \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

(iii) $h(x) = \begin{cases} x \sin \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

Solution

(i) The tangent function f(x) = tanx is undefined at x = (2n +1) π/2, n ∈ Z.

At all other points it is continuous, so f(x) = tan x is continuous on each of the open intervals

(i) The tangent function $f(x) = \tan x$ is undefined at $x = (2n+1)\frac{\pi}{2}, n \in Z$.

At all other points it is continuous, so $f(x) = \tan x$ is continuous on each of the open intervals

$$\ldots\left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right), \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \left(\frac{\pi}{2}, \frac{3\pi}{2}\right), \ldots$$

(ii) The function $y = \frac{1}{x}$ is continuous at all points of $\mathbb{R}$ except at $x = 0$ where it is undefined.

The function $g(x) = \sin\frac{1}{x}$ is continuous at all points except $x = 0$, where $\lim_{x\to 0} g(x)$ does not exist. So, g is continuous on the intervals $(-\infty, 0)$ and $(0, \infty)$

(iii) The function $h(x)$ is defined at all points of the real line $\mathbb{R} = (-\infty, \infty)$; for any $x_0 \neq 0$,

$$\lim_{x\to x_0} h(x) = \left(\lim_{x\to x_0} x\sin\frac{1}{x}\right)$$

$$= x_0 \sin\frac{1}{x_0} = h(x_0)$$

For $x_0 = 0$

$$h(x) = x.\sin\frac{1}{x}$$

$$-x \leq x\sin\frac{1}{x} \leq x$$

$$g(x) = -x, f(x) = x\sin\frac{1}{x}, h(x) = x$$

$$\lim_{x\to 0} g(x) = 0, \lim_{x\to 0} h(x) = 0$$

and have $\lim_{x\to 0} x\sin\frac{1}{x} = 0.$

By Sandwich theorem

$$\lim_{x\to 0}\left(x\sin\frac{1}{x}\right) = 0 = h(0).$$

Therefore $h(x)$ is continuous in the entire real line.

Example 9.38

A tomato wholesaler finds that the price of a newly harvested tomatoes is Rs. 0.16 per kg if he purchases fewer than 100 kgs each day. However, if he purchases at least 100 kgs daily, the price drops to Rs. 0.14 per kg. Find the total cost function and discuss the cost when the purchase is 100 kgs.

Solution

Solution

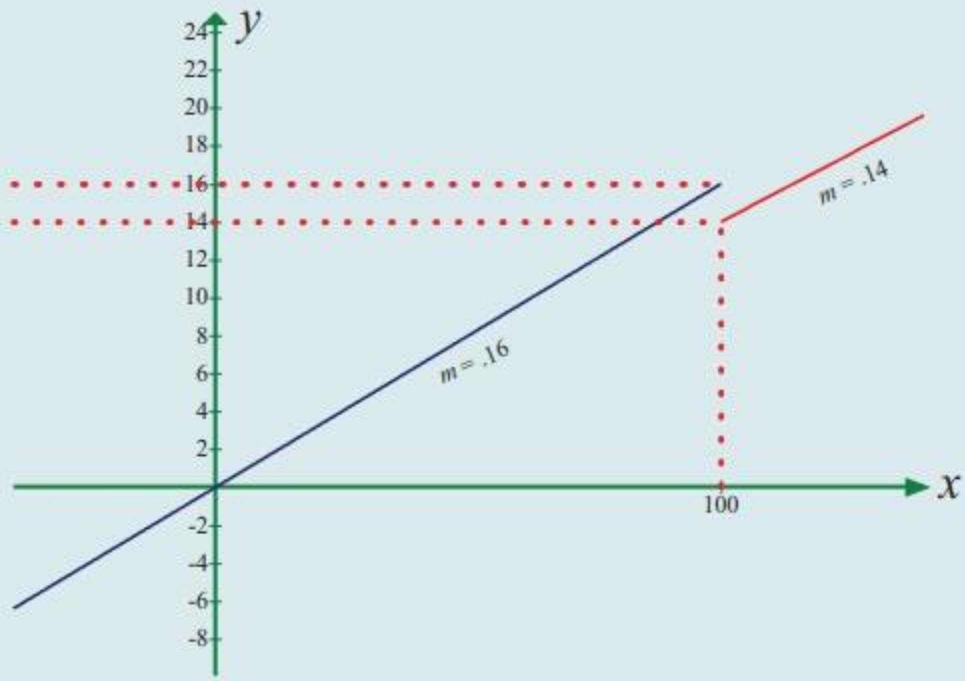


Fig.9.37

Let x denote the number of kilograms bought per day and C denote the cost. Then,

$$C(x) = \begin{cases} 0.16x, & \text{if } 0 \le x < 100 \\ 0.14x, & \text{if } x \ge 100 \end{cases}.$$

The sketch of this function is shown in Fig. 9.37.

It is discontinuous at $x = 100$ since $\lim\limits_{x \to 100^-} C(x) = 16$ and $\lim\limits_{x \to 100^+} C(x) = 14$.

Note that $C(100) = 14$. Thus, $\lim\limits_{x \to 100^-} C(x) = 16 \neq 14 = \lim\limits_{x \to 100^+} C(x) = C(100)$.

Note also that the function jumps from one finite value 14 to another finite value 16.

Continuity of composite functions

Recently, I had the opportunity to evaluate a colleagues' classroom presentation and the topic of the day was continuity of continuous functions. Partway through the presentation of the proof that the composition of continuous functions is continuous, a student asked for a 'picture' of how the ε 's and δ 's were related. The instructor, who happens to be an excellent analyst, confessed that he did not have knowledge of such an illustration. I later found it was not difficult to make use of a graphical technique for function composition [1] to illustrate this proof.

Continuity of a function is a fundamental concept that either directly or indirectly is addressed early in the mathematical education of our students. The idea the graph of a function being connected is explored as soon as students begin to graph simple functions such as lines in the plane. Later, students increase their understanding of continuity as they explore rational functions and trigonometric functions. Formal definitions of continuity are typically introduced in introductory calculus textbooks [for example 4, 5]. These $\varepsilon - \delta$ definitions are typically graphically illustrated at this level; however, some instructors go past that and educate their students at a level where the definition is used explicitly. For many students this higher level of instruction is reserved for an introductory real analysis or advanced calculus course. At all levels, illustrations can be of benefit to the student.

I have now used a graphical technique several times in first semester calculus courses to indicate how the proof would work. The technique has also been applied in more advanced courses. In all cases, students seem to understand the result much better than when they are instructed without the illustration. After a thorough search, I am convinced that this type of illustration does not appear in literature associated with calculus, advanced calculus, or introductory analysis.

Graphical Composition

As in (Davis 2000), let $(f g)(x) = f(g(x))$ denote the composition of two real valued functions $f(x)$ and $g(x)$. In order to evaluate $f(g(x))$ graphically for the value $x = a$:

On the same set of axes draw the graphs of $y = f(x)$, $y = g(x)$, and $y = x$.

Draw a vertical line from the point $x = a$ on the x-axis to the point $(a, g(a))$ on the graph of $y = g(x)$.

Draw a horizontal line from $(a, g(a))$ to the point $(g(a), g(a))$ on the line $y = x$.

Draw a vertical line from $(g(a), g(a))$ to the point $(g(a), f(g(a)))$ on the graph of $y = f(x)$.

Draw a horizontal line from $(g(a), f(g(a)))$ to the point $(0, f(g(a)))$ on the y-axis.

Continuity of Composed Functions

A typical proof of the continuity of composed functions is as follows [for example 2, 3]

Since g is continuous at a , g(a) is defined; likewise, f is continuous at g(a) , so f (g(a)) = (f g)(a) is defined. At this point, using the instructions above, a picture of f (g(a)) is drawn, see Figure 1.

An ε − δ proof is now used to show lim(f g)(x) f g(a)

ε > 0 , δ > 0 must be found to satisfy:

If x − a < δ , then f (g(x)) − f (g(a)) < ε .

Due to the continuity of f at g(a) , we know there is a δ 1

If 1 z − g(a) < δ , then f (z) − f (g(a)) < ε .

Hence when g(x) is within δ 1 of g(a) , then f (g(a)) is within ε of f (g(a)) ; i.e.,

If 1 g(x) − g(a) < δ , then f (g(z)) − f (g(a)) < ε .

Since g is continuous at a , there is a δ > 0 , such that:

If x − a < δ , then 1 g(x) − g(a) < δ .

Chaining inequalities together completes the proof:

x a g x g a 1 f g x f g a .

The Illustration

To draw the corresponding illustration, the following procedure is carried out:

As in Figure 2, draw an ε − neighborhood, on the y-axis, about the point f (g(a)) .

As in Figure 3, draw horizontal lines from the boundaries of this neighborhood to the curve y = f (x).

As in Figure 4, draw two vertical lines from these locations to the line y = x . Since the ε − neighborhood includes f (g(a)) , the vertical lines intersect the line y = x above and below the value of g(a) .

As in Figure 5, draw a δ 1 − neighborhood about g(a) , that when extended to the line y = x , remains inside the region associated with the ε − neighborhood.

As in Figure 6, extend the δ 1 − neighborhood horizontally to the curve y = g(x) .

As in Figure 7, draw horizontal lines from these boundaries to the x-axis. These lines will define an interval that has x = a in its interior.

As in Figure 8, draw a δ − neighborhood about x = a , that remains inside the previous interval.

As in Figure 9, the boundaries of the δ − neighborhood are graphically evaluated by g and then f .

The resulting neighborhood on the y-axes remains within ε of f (g(a)) as desired and the illustration is complete.

Conclusion

I have found that this method of illustrating that the composition of continuous functions is continuous requires no significant additional classroom time. It has proven to be a very helpful teaching aid in the classroom. Students seem to have a better understanding of both function composition as well as the associated continuity properties. I also find that students who have been exposed to this graphical method are more likely to be able to understand how to make choices for ε and/or δ in specific computational exercises. I expect that others will find this type of illustration helpful as well.

Continuity on (a,b) implying boundedness

In the third part of this book, we look more deeply into the properties of functions. We begin in this chapter by considering different ways to define continuity and differentiability and the relations between the different notions. Up to this point, we have employed somewhat restrictive notions of continuity and differentiability in order to make it possible to use constructive arguments to prove major theorems. By considering weaker notions of these concepts, we include more functions in the discussion and also discover some important properties. However, we lose the possibility of using constructive analysis in many cases.

Beginning with this chapter, the discussion takes on a decidedly theoretical flavor and requires more sophistication1 to read. But, a mastery of the material in this part opens up the doors to the entire world of analysis.

A General Notion of Continuity

Recall that the intent in defining Lipschitz continuity was to classify a function as varying smoothly in the sense that small changes in input lead to small changes in output. The Lipschitz continuous condition $|f(x)-f(y)| \leq L|x - y|$ quantifies the maximum amount a function's value can change for a given change in input. We based the notion of Lipschitz continuity on the behavior of linear functions.

But Lipschitz continuity is not the most general way to express the idea that f should vary smoothly

Example 32.1. Consider x1/3, which is Lipschitz continuous on any bounded interval that is bounded away from 0. Checking the Lipschitz condition at 0 gives

|x1/3 − 01/3| = |x| 1/3.

For any constant L, |x| 1/3 > L

|x| for all x sufficiently small;

hence x1/3 cannot be Lipschitz continuous on any interval that contains 0 or has 0 as an endpoint.

On the other hand, |x| 1/3 can be made as close to 0 as desired by making |x| small. So x1/3 does vary smoothly as x passes by 0. We can see this from the plot Fig. 32.1.

We make a general definition of continuity that covers such cases.2 We say that f is continuous at ¯x if given any sufficiently small > 0 there is a δ > 0 such that

|f(x) − f(¯x)| < for all x with |x − x¯| < δ.

In words, this says that the change in value of f(x) from f(¯x) can be made arbitrarily small by taking x sufficiently close to ¯x. Note that f(x) needs to be defined for all x sufficiently close to ¯x. Note also that δ =
δx,

¯ usually depends on both ¯x and .

Unit - III

Sequence

Algebra at the JEE level is very interesting. All topics are more or less independent of each other. And one of the interesting and important topics is Sequences and Series and every year you will get 1 - 2 question in JEE Main exam as well as in other engineering entrance exams. JEE question paper is highly unpredictable, you never know questions from which topic will be asked. A general trend noticed in Mathematics paper is that a question involving multiple concepts are asked. For instance, you will find that questions from Calculus, Matrices and Determinant and Functions where concepts of Sequences and Series are involved. As compared to other chapters in maths, Sequences and Series requires less effort to prepare for the examination.

Why Sequences and Series

Let’s start with one ancient story.
There was a con man who made chessboards for the emperor. The craftsman was good at his work as well as with his mind. He knew that the emperor loved chess. So he conspires a plan to trick the emperor to give him a large amount of fortune. When the craftsman presented his chessboard at court, the emperor was so impressed by the chessboard, that he said to the craftsman

"Name your reward"
The craftsman responded
"Your Highness, I don't want money for this. Or jewels. my wish was simple. All I want is a little rice."
The emperor agreed, amazed that the man had asked for such a small reward
"I've got rice. How much rice?"
"All I want," said the craftsman, "is for you to put a single grain of rice on the first square, two grains on the second square, four on the third square, eight on the fourth square, and so on and so on for all 64 squares, with each square having double the number of grains as the square before."
"Well, I can do that," said the emperor, not thinking much. And he ordered his treasurer to pay the craftsman for the chessboard.

Well, that turned out to be more than a little difficult. The first few squares on the board cost the emperor 1 grain, then 2, then 4 ... by the end of the first row, he was up to 128 grains.
In the second row, things got out of control. By the 21st square he owed over a million grains of rice; by the 41st, it was over a trillion grains of rice — more rice than he, his subjects or any emperor anywhere could afford in the world.
After all, he was the emperor. He knew how to handle such situations
"I will pay you," he told the craftsman. "But before you receive the grains of rice, just to be sure you are getting what you asked for, I'd like you to count each and every grain I give you."
"Oh, that won't be required," said the craftsman.
"Oh, it is necessary," said the emperor. "I wouldn't want to cheat you."
So now you tell me, What will be the total number of grains? How much time does craftsman require to complete the count? The amount of rice that craftsman asked, will that be available on our planet?
Well, all the answers to these questions you will able to tell when you study Sequences and Series.

After reading this chapter you will be able to:

- Write the first few terms of a sequence
- Find a formula for the general term (nth term) of a sequence
- Find the sum and partial sum
- Use of summation notation to write a sum

Important Topics

- Sequences
- Arithmetic and Geometric Progression
- Arithmetic and Geometric Mean
- Harmonic Progression
- Sum up to n terms
- Arithmetic-geometric series

Overview of Chapter- Sequence and Series

Sequences

A sequence is an arrangement of a list of objects or numbers in a definite order. The numbers or objects are also known as the terms of the sequence. A sequence containing a finite number of terms is called a finite sequence and a sequence is called infinite if it is not a finite sequence.

eg.. 2, 4, 6, 8, 10, 12, …..

Often when working with sequences we do not want to write out all the terms. We want a more compact way to show how each term is defined. And hence, we define the general term and it is denoted by a_n.or t_n.

For the above example, $a_n = 2n$, where n is Integer.

Arithmetic Progression

An arithmetic progression is a sequence whose terms increase or decrease by a fixed number. The fixed number is called a common difference (d) of the AP. If a is the first term, and d is a common difference, then AP can be written as

a, a + d, a + 2d, a + 3d,   ……….. a + (n - 1)d

Where a + (n - 1)d is the general term of an AP.

Sum of n terms of an AP is given by

$$s_n = \frac{n}{2}\left[2a + (n-1)d\right]$$

You know that the sum of the interior angle is 180^0, of a quadrilateral, is 360^0 and of a Pentagon is 540^0 . Assume that the patterns continue. Then the sum of the

interior angle of an Octagon (8 sided) is

The pattern 180^0, 360^0, 540^0 …….is arithmetic with common difference 180^0. The 8-sided figure will be the 6th term in the sequance.

6th term is

$a_6 = a_1 + (n - 1)(d)$

$a_6 = 180^0 + (6 - 1)(180^0) = 1080^0$

Geometric Progression

A geometric sequence is a sequence such that if the ratio of any term and its just preceding term is constant throughout. The onstant called the common ratio which is denoted by r.

$$r = \frac{a_2}{a_1} = \frac{a_3}{a_2} = \ldots\ldots.. = \frac{a_n}{a_{n-1}}$$

Where,

r = common ratio

a_1 = first term

a_2 = second term

a_n = nth term

Important Formula of Geometric Progression

1. The general term of a GP is $a_n = ar^{n-1}$
2. Sum of n-term of a geometric progression is given by $s_n = \frac{a_1(r^n - 1)}{r - 1}$
3. If geometric progression is infinite and common ratio, -1 < r < 1, then the sum of the series is given by $s_n = \frac{a}{1 - r}$

Now, consider the earlier story

The first square contains 1 rice, the second square contains 2 rice grains, the 3rd square contains 4 rice grains, 4th one contains 8 rice grains and so on…
The sequence will be like 1, 2, 4, 8, 16, 32,......... or $\{2^0, 2^1, 2^3, 2^4 \ldots\ldots..2^{63}\}$, this is a geometric progression,
Using the formula of summation of a geometric progression

Which is weighing about 1,199,000,000,000 metric tons (assuming 65 mg as the mass of one grain of rice)
It would have taken the craftsman a half-trillion years, about 42 times the age of our universe, to complete his count.

How to prepare Sequences and Series?

Sequences and series is one of the easiest topics, you can prepare this topic without applying many efforts

- Start with basic theory, understand all the definition of the Sequences, series, and Arithmetic and geometric progression.

- Derive and understand the formulae of General Term, Sum of the Series of n terms and remember standard results.

- Learn the concept behind Harmonic sequences and general term of Harmonic sequences.

- Derive all the formulae of summation of some special series like the sum of first n natural number, the summation of odd numbers, sum of cube of first n natural numbers, etc.

- Mean is one of the most important concepts, as AM-GM is used to determine the minimum and maximum value of the function.
- After the study, each concept, do a lot of solved examples in order to comprehend the concept as well as their applications.
- Make sure that after studying certain section/concept, solve questions related to those concepts without looking into the solutions and practice MCQ from the above-mentioned books and solve all the previous year problems asked in JEE.
- Don't let any doubt remain in your mind and clear all the doubts with your teachers or with your friends.

Best Books For Preparation:-

First, finish all the concept, example and question given in NCERT Maths Book. You must thorough with the theory of NCERT. Then you can refer to the book Cengage Mathematics Algebra. Sequences and Series are explained very well in this book and there are ample amount of questions with crystal clear concepts. You can also refer to the book Arihant Algebra by SK Goyal or RD Sharma. But again the choice of reference book depends on person to person, find the book that best suits you the best depending on how well you are clear with the concepts and the difficulty of the questions you require.

convergent sequence

Definition 1. A sequence of real numbers (s_n) is said to converge to a real number s if

$\forall \varepsilon > 0, \exists N \in \mathbb{N}$, such that $n > N$ implies $|s_n - s| < \varepsilon$.

When this holds, we say that (s_n) is a convergence sequence with s being its limit, and write $s_n \to s$ or $s = \lim_{n\to\infty} s_n$. If (s_n) does not converge, then we say that (s_n) is a divergent sequence.

We first show that one sequence (s_n) can not have two different limits. Suppose $s_n \to s$ and $s_n \to t$. Let $\varepsilon > 0$. Then $\frac{\varepsilon}{2} > 0$. Since $s_n \to s$, by definition there is $N_1 \in \mathbb{N}$ such that for $n > N_1$, $|s_n - s| < \frac{\varepsilon}{2}$. Since $s_n \to t$, by definition there is $N_2 \in \mathbb{N}$ such that for $n > N_2$, $|s_n - t| < \frac{\varepsilon}{2}$. Here we use N_1 and N_2 in the two statements because the N coming from the two limits may not be the same. Let $N = \max\{N_1, N_2\}$. If $n > N$, then $n > N_1$ and

$n > N_2$ both hold. So $|s_n - s| < \varepsilon\ 2$ and $|s_n - t| < \varepsilon\ 2$, which by triangle inequality imply that

$|s - t| \leq |s_n - s| + |s_n - t| < \varepsilon\ 2 + \varepsilon\ 2 = \varepsilon$.

Now $|s-t| < \varepsilon$ holds for every $\varepsilon > 0$. We then conclude that $|s-t| = 0$ (for otherwise $|s-t| > 0$, we then get a contradiction by choosing $\varepsilon = |s - t|$). So $s = t$, and the uniqueness holds. We will use the following tools to check whether a sequence converges or diverges.

1. the definition

2. basic examples

3. limit theorems

4. boundedness and subsequences.

We have stated the definition. Now we consider some examples.

Example 1. Let $s \in R$. If $s_n = s$ for all n, i.e., (s_n) is a constant sequence, then $\lim s_n = s$. Proof. For any given $\varepsilon > 0$ we simply choose $N = 1$. If $n > N$, then $|s_n - s| = 0 < \varepsilon$. Example 2. We have $1\ n \to 0$.

Proof. Let $\varepsilon > 0$. By Archimedean property, there is $N \in N$ such that $1\ N < \varepsilon$. If $n > N$, then

Example 3. The following two sequences are divergent

(i) $(s_n) = ((-1)^n) = (-1, 1, -1, 1, -1, 1, \ldots)$;

(ii) $(s_n) = (n) = (1, 2, 3, 4, 5, 6, \ldots)$.

Proof. (i) We use the notation of subsequence and statement that will be proved later. Suppose $n_1 < n_2 < n_3 < \cdots$ is a strictly increasing sequence of indices, then (s_{n_k}) is a subsequence of (s_n). We will prove a theorem, which asserts that, if (s_n) converges to s, then any subsequence of (s_n) also converges to s. The sequence $(s_n) = ((-1)^n)$ contains two constant sequences $(1, 1, 1, \ldots)$ (with $n_k = 2k$) and $(-1, -1, -1, \ldots)$ (with $n_k = 2k-1$), which converge to different limits. So the original (s_n) can not converge.

(ii) We use the following theorem. If (s_n) is convergent, then it is a bounded sequence. In other words, the set $\{s_n : n \in N\}$ is bounded. So an unbounded sequence must diverge. Since for $s_n = n$, $n \in N$, the set $\{s_n : n \in N\} = N$ is unbounded, the sequence (n) is divergent.

Remark 1. This example shows that we have two ways to prove that a sequence is divergent: (i) find two subsequences that convergent to different limits; (ii) show that the sequence is unbounded. Note that the (s_n) in (i) is bounded and divergent. The (s_n) in (ii) is divergent, but $\lim s_n$ actually exists, which is $+\infty$, and its every subsequence also tends to $+\infty$. We will define that limit later.

Now we state some limit theorems.

Theorem 1 (Theorem 9.1). Every convergent sequence is bounded.

Proof. Let (s_n) be a sequence that converges to $s \in \mathbb{R}$. Applying the definition to $\varepsilon = 1$, we see that there is $N \in \mathbb{N}$ such that for any $n > N$, $|s_n - s| < 1$, which then implies that $|s_n| \le |s| + 1$. Let

$$M = \max\{|s_1|, |s_2|, \ldots, |s_N|, |s| + 1\}.$$

The maximum exists since the set is finite. Then for any $n \in \mathbb{N}$, $|s_n| \le M$ (consider the case $n \le N$ and $n > N$ separately), i.e., $-M \le s_n \le M$. So $\{s_n : n \in \mathbb{N}\}$ is bounded.

Theorem 2 (Theorem 9.3). If (s_n) converges to s and (t_n) converges to t, then $(s_n + t_n)$ converges to $s + t$.

Proof. Let $\varepsilon > 0$. Then $\frac{\varepsilon}{2} > 0$. Since $s_n \to s$, there is $N_1 \in \mathbb{N}$ such that for $n > N_1$, $|s_n - s| < \frac{\varepsilon}{2}$. Since $t_n \to t$, there is $N_2 \in \mathbb{N}$ such that for $n > N_2$, $|t_n - t| < \frac{\varepsilon}{2}$. Let $N = \max\{N_1, N_2\}$. If $n > N$, then $n > N_1$ and $n > N_2$ both hold, and so $|s_n - s| < \frac{\varepsilon}{2}$ and $|t_n - t| < \frac{\varepsilon}{2}$, which together imply (by triangle inequality) that

$$|(s_n + t_n) - (s + t)| \le |s_n - s| + |t_n - t| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

So we have the desired convergence.

Theorem 3 (Theorem 9.4). If (s_n) converges to s and (t_n) converges to t, then $(s_n \cdot t_n)$ converges to $s \cdot t$.

Discussion. We need to bound $|s_n t_n - st|$ from above for big n. We write

$$s_n t_n - st = s_n t_n - s_n t + s_n t - st = s_n(t_n - t) + t(s_n - s).$$

By triangle inequality, we get

$$s_n t_n - st| \le |s_n(t_n - t)| + |t(s_n - s)| = |s_n||t_n - t| + |t||s_n - s|.$$

Since $t_n \to t$ and $s_n \to s$, we know that $|t_n - t|$ and $|s_n - s|$ can be arbitrarily small if we choose n big enough. Thus, if $|s_n|$ and $|t|$ are not too big, then we can control the sum on the RHS (righthand side). In fact, the size of $|s_n|$ can be controlled because of Theorem 9.1.

Proof. Since (s_n) is convergent, by Theorem 9.1, there is $M > 0$ such that $|s_n| \leq M$ for every n. We may choose M big such that $M \geq |t|$. Let $\varepsilon > 0$. Then ε 2M > 0. Since $s_n \to s$, there is $N1 \in N$ such that for $n > N1$, $|s_n - s| < \varepsilon$ 2M . Since $t_n \to t$, there is $N2 \in N$ such that for $n > N2$, $|t_n - t| < \varepsilon$ 2M . Let $N = \max\{N1, N2\}$. If $n > N$, then $n > N1$ and $n > N2$ both hold, and so $|s_n - s| < \varepsilon$ 2M and $|t_n - t| < \varepsilon$ 2M , which together with $|s_n| \leq M$ and $|t| \leq M$ imply that

$s_nt_n - st| \leq |s_n(t_n - t)| + |t(s_n - s)| = |s_n||t_n - t| + |t||s_n - s|$

$\leq M|t_n - t| + M|s_n - s| < M\,\varepsilon\,2M + M\,\varepsilon\,2M = \varepsilon.$

Corollary 1. If (s_n) converges to s, $k \in R$, and $m \in N$, then (ks_n) converges to ks and s m n converges to s m.

Proof. For the sequence (ks_n), we apply Theorem 9.4 to the sequence (t_n) with $t_n = k$ for all n. For the sequence (s m n) we use induction. In the induction step, note that s m+1 n = $s_n \ast$ s m n and apply Theorem 9.4 to t_n = s m n

Corollary 2. If (s_n) converges to s and (t_n) converges to t, then $(s_n - t_n)$ converges to s − t. Proof. We write $s_n + t_n = s_n + (-1)t_n$ and apply Theorem 9.3 and the previous corollary.

From this corollary we see that $s_n \to s$ iff $s_n - s \to 0$. By the Theorem below, the latter statement is equivalent to that $|s_n - s| \to 0$.

Theorem 4. (a) Suppose two sequences (s_n) and (t_n) satisfy that $t_n \to 0$ and $|s_n| \leq |t_n|$ for all but finitely many n. Then $s_n \to 0$.

(b) For any sequence (s_n), $s_n \to 0$ if and only if $|s_n| \to 0$.

Proof. (a) Let $N0 \in N$ be such that $|s_n| \leq |t_n|$ for $n > N0$. Let $\varepsilon > 0$. Since $t_n \to 0$, there is $N1 \in N$ such that for $n > N1$, $|t_n - 0| < \varepsilon$. Let $N = \max\{N0, N1\}$. For $n > N$, $|s_n| \leq |t_n|$ and $|t_n - 0| < \varepsilon$, which imply that $|s_n - 0| = |s_n| \leq |t_n| = |t_n - 0| < \varepsilon$.

(b) From (a) we know that if $|s_n| = |t_n|$ for all n, then $s_n \to 0$ iff $t_n \to 0$. We then apply this result to $t_n = |s_n|$ and use that $||s_n|| = |s_n|$.

Lemma 1 (Lemma 9.5). If (s_n) converges to s such that s 6= 0 and s_n 6= 0 for all n, then $(1/s_n)$ converges to 1/s.

Discussion. We need to bound $|1/s_n - 1/s|$ from above for big n. We write

Since $s_n \to s$, $|s_n - s|$ can be arbitrarily small if we choose n big enough. Thus, if $|s_n|$ and $|s|$ are not too close to 0, then we can control the size of the RHS. This means that we need a positive lower bound of the set $\{|s_1|, |s_2|, \ldots\}$.

Proof. Since s 6= 0, we have |s| 2 > 0. Since sn → s, applying the definition to ε = |s| 2 , we get N ∈ N such that for n > N, |sn − s| < |s| 2 , which then implies by triangle inequality that |sn| ≥ |s| − |sn − s| > |s| − |s| 2 = |s| 2 . Let m = min{|s1|, |s2|, . . . , |sN |, |s| 2 }. Then m exists and is positive since the set is a finite set of positive numbers.

Let ε > 0. Then m|s|ε > 0. Since sn → s, there is N0 ∈ N such that n > N0 implies that |sn − s| < m|s|ε, which together with |sn| ≥ m for all n implies that

Theorem 5 (Theorem 9.6). Suppose (sn) converges to s and (tn) converges to t. If s 6= 0 and sn 6= 0 for all n, then (tn/sn) converges to t/s.

Proof. By Lemma 9.5, (1/sn) converges to 1/s. Applying Theorem 9.4 to the sequences (1/sn) and (tn), we get the conclusion.

Cauchy Sequence

One of the problems with deciding if a sequence is convergent is that you need to have a limit before you can test the definition.

Bernard Bolzano was the first to spot a way round this problem by using an idea first introduced by the French mathematician Augustin Louis Cauchy (1789 to 1857).

Definition

> A sequence is called a **Cauchy sequence** if the terms of the sequence eventually all become arbitrarily close to one another.
> That is, given ε > 0 there exists N such that if m, n > N then $|a_m - a_n| < ε$.

Remarks

1. Note that this definition does not mention a limit and so can be checked from knowledge about the sequence.

2. It is not enough to have each term "close" to the next one. ($|a_m - a_{m+1}| < ε$. For example, the divergent sequence of partial sums of the harmonic series (see this earlier example) does satisfy this property, but not the condition for a Cauchy sequence.

3. We will see (shortly) that Cauchy sequences are the same as convergent sequences for sequences in **R**. However, we will see later that when we introduce the idea of convergent in a more general context Cauchy sequences and convergent sequences may be different.

4. Cantor (1845 to 1918) used the idea of a Cauchy sequence of rationals to give a constructive definition of the Real numbers independent of the use of Dedekind Sections.

Some properties of Cauchy sequences

1. Any Cauchy sequence is bounded.

   Proof
   (When we introduce Cauchy sequences in a more general context later, this result will still hold.)
   The proof is essentially the same as the corresponding result for convergent sequences.

2. Any convergent sequence is a Cauchy sequence.

   Proof
   If $(a_n) \rightarrow \alpha$ then given $\varepsilon > 0$ choose N so that if $n > N$ we have $|a_n - \alpha| < \varepsilon$. Then if $m, n > N$ we have $|a_m - a_n| = |(a_m - \alpha) - (a_m - \alpha)| \leq |a_m - \alpha| + |a_m - \alpha| < 2\varepsilon$.

3. **The Main Result about Cauchy sequences**

   A Real Cauchy sequence is convergent.

   Proof
   Since the sequence is bounded it has a convergent subsequence with limit α.
   Claim:
   This α is the limit of the Cauchy sequence.
   Proof of that:
   Given $\varepsilon > 0$ go far enough down the subsequence that a term a_n of the subsequence is within ε of α. Provided we are far enough down the Cauchy sequence any a_m will be within ε of this a_n and hence within 2ε of α.

Remarks

1. The fact that in **R** Cauchy sequences are the same as convergent sequences is sometimes called the Cauchy criterion for convergence.

2. The use of the Completeness Axiom to prove the last result is crucial. For example, let (a_n) be a sequence of rational numbers converging to an irrational. [e.g. (1, 1.4, 1.41, 1.414, ...)→ √2]

   Then since (a_n) is a convergent sequence in **R** it is a Cauchy sequence in **R** and hence also a Cauchy sequence in **Q**. But it has no limit in **Q**.

3. In fact one can formulate the Completeness axiom in terms of Cauchy sequences.

   Here are some equivalent formulations of the axiom

   III Every subset of **R** which is bounded above has a least upper bound.

   III* In **R** every bounded monotonic sequence is convergent.

   III** In **R** every Cauchy sequence is convergent.

   We will see later that the formulation **III**** is a useful way of generalising the idea of completeness to structures which are more general than ordered fields.

monotonic sequence

Definition : We say that a sequence (xn) is increasing if xn ≤ xn+1 for all n and strictly increasing if xn < xn+1 for all n. Similarly, we define decreasing and strictly decreasing sequences. Sequences which are either increasing or decreasing are called monotone.

The following result is an application of the least upper bound property of the real number system

Theorem 2.5: Suppose (xn) is a bounded and increasing sequence. Then the least upper bound of the set {xn : n ∈ N} is the limit of (xn).

Proof: Suppose sup n xn = M. Then for given > 0, there exists n0 such that M − ≤ xn0 . Since (xn) is increasing, we have xn0 ≤ xn for all n ≥ n0. This implies that

M − ≤ xn ≤ M ≤ M + for all n ≥ n0.

That is xn → M.

For decreasing sequences we have the following result and its proof is similar.

Theorem 2.6: Suppose (xn) is a bounded and decreasing sequence. Then the greatest lower bound of the set {xn : n ∈ N} is the limit of (xn).

Examples: 1. Let x1 = √ 2 and xn = √ 2 + xn−1 for n > 1. Then use induction to see that 0 ≤ xn ≤ 2 and (xn) is increasing. Therefore, by previous result (xn) converges. Suppose xn → λ. Then λ = √ 2 + λ. This implies that λ = 2.

2. Let x1 = 8 and xn+1 = 1 2 xn + 2. Note that xn+1 xn < 1. Hence the sequence is decreasing. Since xn > 0, the sequence is bounded below. Therefore (xn) converges. Suppose xn → λ. Then λ = λ 2 + 2. Therefore, λ = 2.

Sub-sequence

We have seen some bounded sequences which do not converge. We can, however, say something about such sequences.

Definition

A **subsequence** is an infinite ordered subset of a sequence.

Examples

$(a_2 , a_4 , a_6 , \dots)$ is a subsequence of $(a_1 , a_2 , a_3 , a_4 , \dots)$. So is $(a_1 , a_{10} , a_{100} , a_{1000} , \dots)$.

Theorem

Any subsequence of a convergent sequence is convergent (to the same limit).

Proof
Look at the definition!

□

The nicest thing about these subsequences is a result attributed to the Czech mathematician and philosopher Bernard Bolzano (1781 to 1848) and the German mathematician Karl Weierstrass (1815 to 1897).

The Bolzano-Weierstrass Theorem

Every bounded sequence has a convergent subsequence.

Remark

Notice that a bounded sequence may have many convergent subsequences (for example, a sequence consisting of a counting of the rationals has subsequences converging to every real number) or rather few (for example a convergent sequence has all its subsequences having the same limit).

Proof
Suppose the sequence (a_1 , a_2 , a_3 , a_4 , ...) is bounded and lies in (say) the interval [0, 1].

Then we construct a convergent subsequence by a bisection process.
Split the interval [0, 1] into two halves [0 , $^1/_2$] and [$^1/_2$, 1]. Then (at least) one of the halves will contain infinitely many terms of the sequence. Suppose it is [0 , $^1/_2$].
Choose $x_1$to be one of these terms.
Then split this interval in half again and repeat the process choosing x_2 to be further down the sequence than x_1. Continuing in this way, at the nth stage we will choose a term x_n lying in an interval [a_n , b_n].

Claim: The subsequence (x_n) is convergent.

Proof of claim

The sequence (l_n) of "Left-hand ends" of intervals is monotonic increasing, bounded above by 1 and hence has a limit α.

The sequence (r_n) of "right-hand ends" of intervals is monotonic decreasing, bounded below by 0 and hence has a limit β.

Since the length of the interval [l_n , r_n] has length $(^1/_2)^n$, we must have α = β and since the sequence (x_n) is trapped between (l_n) and (r_n), it converges to the same limit.

Limit superior and limit inferior of sequences

A number a is called a limit point of the sequence {an} if it is the limit of a subsequence of {an}. A bounded sequence has at least one limit point according to Bolzano-Weierstrass Theorem. A properly divergent sequence does not have any limit point.

Let {an} be a sequence bounded from below. For each k ≥ 1, the number

βk = sup n≥k an = {ak, ak+1, ak+2, · · · } ,

is in (−∞, ∞]. It is clear that {βk} is decreasing and bounded from below. By Monotone Convergence Theorem, its limit exists. We call it the limit superior of the sequence of {an}. In notation,

lim an, or lim sup{an} = lim k→∞ βk = inf{βk} = inf k sup{an}n≥k .

Similarly, the number

αk = inf n≥k an = inf{ak, ak+1, ak+2, · · · } , is a real number when the sequence is bounded from above. It is clear that $\{\alpha_k\}$ is increasing and bounded from above. By Monotone Convergence Theorem, its limit exists. We call it the limit inferior of the sequence of $\{a_n\}$. In notation,

$\lim a_n$, or $\lim\inf\{a_n\} = \lim_{k\to\infty} \alpha_k = \sup\{\alpha_k\} = \sup_k \inf\{a_n\}_{n\geq k}$.

Theorem 1. Let $a = \lim_{n\to\infty} a_n$.

(a) For each $\varepsilon > 0$, there is some n_0 such that $a_n \leq a + \varepsilon$ for all $n \geq n_0$.

(b) For each $\varepsilon > 0$, there is a subsequence $\{a_{n_j}\}$ satisfying $a_{n_j} \geq a - \varepsilon$.

Proof. (a) By the definition of infimum, for any $\varepsilon > 0$, there is some k_0 such that $\beta_k \leq \alpha + \varepsilon$ for all $k \geq k_0$. It follows from the definition of β_k that $a_n \leq a + \varepsilon$ for all $n \geq k_0$. It suffices to take $n_0 = k_0$.

(b) It suffices to show there is a subsequence converging to a. Since $a = \lim_{k\to\infty} \beta_k = \inf_k \beta_k$, for each $N \geq 1$, there is some $n(N)$ such that

$a + 1 N > \beta_{n(N)} \geq a$.

From the definition of the supremum, we can find a_{n_N} from $\{a_{n(N)}, a_{n(N)+1}, a_{n(N)+2}, \cdots\}$ to form a subsequence $\{a_{n_N}\}$ such that

$\beta_{n(N)} \geq a_{n_N} > \beta_{n(N)} - 1 N$.

Combining (1) and (2), we have

$|a_{n_N} - a| < 1 N$.

It follows that the subsequence $\{a_{n_N}\}^\infty_{N=1}$ converges to a.

From Theorem 1, we deduce the following characterization of limit superior and limit inferior

Theorem 2. The limit superior of a bounded sequence is its largest limit point and its limit infimum is its smallest limit point.

As an application to power series, we prove

Theorem 3 (Cauchy-Hadamard) The power series $P a_n x^n$ is absolutely and uniformly convergent on $[-r, r]$ for $r \in (0, R)$ where R is its radius of convergence, and it is divergent at any x, $|x| > R$.

We have taken the center $x_0 = 0$ for simplicity. Recall that a series of functions $P f_n$ is called absolutely and uniformly convergent on some set E if $P^\infty_{k=1} |f_k|(x)$ is uniformly convergent on E. It implies that $P^\infty_{k=1} f_k(x)$ is also uniformly convergent on E.

Proof. Recall that $R = 1/\rho$ where $\rho = \lim_{n\to\infty}|a_n|^{1/n} \in [0, \infty]$. According to Theorem 1(a), for each $\varepsilon > 0$, $|a_n|^{1/n} \le \rho + \varepsilon$ for all $n \ge n_0$. As a result,

$$(|a_n||x|^n)^{1/n} = |a_n|^{1/n}|x| \le r|a_n|^{1/n} \le r(\rho + \varepsilon), \ \forall x \in [-r, r], \ n \ge n_0 .$$

Observing that $r(\rho+\varepsilon) < 1$ when $\varepsilon = 0$, we can find a small $\varepsilon_0 > 0$ such that $r_0 \equiv r(\rho+\varepsilon_0) < 1$. It follows that

$$a_n||x|^n \le r_0^n, \ \forall n \ge n_0 .$$

By M-Test, $\sum a_n x^n$ converges absolutely and uniformly on $[-r, r]$.

On the other hand, for each $\varepsilon > 0$, there is a subsequence a_{n_n} satisfying $a_{n_j} \ge a - \varepsilon$. Therefore, $|a_n x^n|^{1/n} = |x||a_n|^{1/n} \ge |x|(\rho - \varepsilon)$ at all $n = n_j$. Since $|x|\rho > 1$, we can fix a small ε_1 such that $|x|(\rho - \varepsilon_1) \ge 1$, so $|a_n x^n| \ge 1$ at all $n = n_j$. It implies that $\sum a_n x^n$ is divergent (since $a_n x^n$ must converge to 0 when it is convergent).

Unit - IV

Infinite series

Infinite series, the sum of infinitely many numbers related in a given way and listed in a given order. Infinite series are useful in mathematics and in such disciplines as physics, chemistry, biology, and engineering.

For an infinite series $a_1 + a_2 + a_3 + \cdots$, a quantity $s_n = a_1 + a_2 + \cdots + a_n$, which involves adding only the first n terms, is called a partial sum of the series. If s_n approaches a fixed number S as n becomes larger and larger, the series is said to converge. In this case, S is called the sum of the series. An infinite series that does not converge is said to diverge. In the case of divergence, no value of a sum is assigned. For example, the nth partial sum of the infinite series $1 + 1 + 1 + \cdots$ is n. As more terms are added, the partial sum fails to approach any finite value (it grows without bound). Thus, the series

diverges. An example of a convergent series is

$$1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n}$$

As n becomes larger, the partial sum approaches 2, which is the sum of this infinite series. In fact, the series $1 + r + r^2 + r^3 + \cdots$ (in the example above r equals 1/2) converges to the sum $1/(1 - r)$ if $0 < r < 1$ and diverges if $r \geq 1$. This series is called the geometric series with ratio r and was one of the first infinite series to be studied. Its solution goes back to Zeno of Elea's paradox involving a race between Achilles and a tortoise (*see* mathematics, foundations of: Being versus becoming).

Certain standard tests can be applied to determine the convergence or divergence of a given series, but such a determination is not always possible. In general, if the series $a_1 + a_2 + \cdots$ converges, then it must be true that a_n approaches 0 as n becomes larger. Furthermore, adding or deleting a finite number of terms from a series never affects whether or not the series converges. Furthermore, if all the terms in a series are positive, its partial sums will increase, either approaching a finite quantity (converging) or growing without bound (diverging). This observation leads to what is called the comparison test: if $0 \leq a_n \leq b_n$ for all n and if $b_1 + b_2 + \cdots$ is a convergent infinite series, then $a_1 + a_2 + \cdots$ also converges. When the comparison test is applied to a geometric series, it is reformulated slightly and called the ratio test: if $a_n > 0$ and if $a_{n+1}/a_n \leq r$ for some $r < 1$ for every n, then $a_1 + a_2 + \cdots$ converges. For example, the ratio test proves the convergence of the series

$$1 + \frac{1}{2} + \frac{1}{3 \cdot 2} + \frac{1}{4 \cdot 3 \cdot 2} + \cdots$$

Many mathematical problems that involve a complicated function can be solved directly and easily when the function can be expressed as an infinite series involving trigonometric functions (sine and cosine). The process of breaking up a rather arbitrary function into an infinite trigonometric series is called Fourier analysis or harmonic analysis and has numerous applications in the study of various wave phenomena.

convergence of series

Convergence, in mathematics, property (exhibited by certain infinite series and functions) of approaching a limit more and more closely as an argument (variable) of the function increases or decreases or as the number of terms of the series increases.

For example, the function $y = 1/x$ converges to zero as x increases. Although no finite value of x will cause the value of y to actually become zero, the limiting value of y is zero because y can be made as small as desired by choosing x large enough. The line $y = 0$ (the x-axis) is called an asymptote of the function.

Similarly, for any value of x between (but not including) −1 and +1, the series $1 + x + x^2 +\cdots+ x^n$ converges toward the limit $1/(1 - x)$ as n, the number of terms, increases. The interval $-1 < x < 1$ is called the range of convergence of the series; for values of x outside this range, the series is said to diverge.

series of positive terms

The comparison test that is considered in this concept is based on the ideas that (1) if a positive term series is always greater, term by term, than another infinite series that diverges, than the positive term series must also diverge, and (2) if a positive term series is always smaller, term by term, than another infinite series that converges, than the positive term series must also converge. If △n represents how much greater or smaller a term an of a series is compared to a term bn of a series of known convergence or divergence, can you formulate expressions that show (1) and (2)?

Comparison Tests

Series with only nonnegative terms, i.e., terms that are either positive or zero, are often called positive term series, and are described as ∑k=1∞uk with uk≥0 for every k. Several of the types of series identified in the previous concepts are, or can be, nonnegative (or positive) term series as shown below:

Common Series Types can be (are) Positive Term Series

Series Type	**Sigma Notation**	**Converges if**	**Diverges if**	**Positive Term Series if**
Arithmetic	$S=\sum_{n=1}^{\infty}[t_0+d(n-1)]$	Never	Always	$t_0,d\geq 0$
Geometric	$S=\sum_{n=1}^{\infty}ar^{n-1}$	$\lvert r\rvert<1$ with $S=\frac{a}{1-r}$	$\lvert r\rvert\geq 1$	$a,r\geq 0$
Harmonic	$S=\sum_{n=1}^{\infty}\frac{1}{n}$	Never	Always	Always
p-Series	$S=\sum_{n=1}^{\infty}\frac{1}{n^p},p>0$	$p>1$	$0<p\leq 1$	Always

So far we have looked at the following tests for the convergence/divergence of infinite series:

Convergence/Divergence Test	**Applicable Series**
Limit of nth partial sum	All
nth-term test for divergence	All
Integral Test	Positive term

The following tests are specifically made for evaluating positive term series:

- The Integral Test
- Comparison Tests (the Basic, The Simplified Limit Comparison Test)
- Ratio and Root Tests

This concept will focus on several comparison tests, i.e. tests that compare one infinite series of unknown convergence with another series of known convergence. The comparison can be term by term, or via the ratio of terms.

The (Direct) Comparison Test

The name of the test tells us that we will compare one series to another to determine convergence or divergence.

The **(Direct) Comparison Test** is as follows:

Suppose $\sum_{k=1}^{\infty} u_k$ and $\sum_{k=1}^{\infty} v_k$ are series with non-negative terms, then:

1. If $u_k \leq v_k$ for every positive integer k and $\sum_{k=1}^{\infty} v_k$ converges, then $\sum_{k=1}^{\infty} u_k$ converges.

2. If $u_k \geq v_k$ for every positive integer k and $\sum_{k=1}^{\infty} v_k$ diverges, then $\sum_{k=1}^{\infty} u_k$ diverges.

In order to use this test, we must check the relationship between u_k and v_k for each index k. This is the comparison part of the test. If the series with the greater-valued terms converges, than the series with the lesser-valued terms converges. If the lesser-valued series diverges, then the greater-valued series will diverge.

Let's determine whether $\sum_{k=1}^{\infty} \frac{1}{k^3+3}$ converges or diverges.

$\sum_{k=1}^{\infty} \frac{1}{k^3+3}$ looks similar to $\sum_{k=1}^{\infty} \frac{1}{k^3}$, so we will try to apply the Comparison Test. First compare each term of both series: for

each k, $\frac{1}{k^3+3} < \frac{1}{k^3}$ so $\sum_{k=1}^{\infty} \frac{1}{k^3+3} < \sum_{k=1}^{\infty} \frac{1}{k^3}$.

Next, we know that $\sum_{k=1}^{\infty} \frac{1}{k^3}$ is a p-series that converges because $p>1$. Therefore, by the Comparison Test, $\sum_{k=1}^{\infty} \frac{1}{k^3+3}$ also converges.

The Limit Comparison Test

The Limit Comparison Test is easier to use than the Comparison Test for determining the convergence of series non-negative terms.

The **Limit Comparison Test** is as follows:

Suppose $\sum_{k=1}^{\infty} u_k$ and $\sum_{k=1}^{\infty} v_k$ are series with $u_k>0$ and $v_k>0$ for all k,
then:

1. If $\lim_{k\to\infty}\frac{u_k}{v_k}=L$, where $0<L<\infty$, then $\sum_{k=1}^{\infty} u_k$ and $\sum_{k=1}^{\infty} v_k$ both converge or both diverge.

2. If $\lim_{k\to\infty}\frac{u_k}{v_k}=0$ and $\sum_{k=1}^{\infty} v_k$ converges, then $\sum_{k=1}^{\infty} u_k$ converges.

3. If $\lim_{k\to\infty}\frac{u_k}{v_k}=+\infty$ and $\sum_{k=1}^{\infty} v_k$ diverges, then $\sum_{k=1}^{\infty} u_k$ diverges.

The Limit Comparison Test says to make a ratio of the terms of two series and compute the limit. Unlike the Comparison Test, there is no need to compare the terms of both series. This test is most useful for series with rational expressions.

Let's apply the limit comparison test and determine

if $\sum_{k=1}^{\infty}\frac{k^4+6k^3-1}{7k^5+k^2}$ converges or diverges.

Just as with rational functions, when $k\to\infty$ the series $\sum_{k=1}^{\infty}\frac{k^4+6k^3-1}{7k^5+k^2}$ behaves like the series with only the highest powers of k in the numerator and denominator:

$\sum_{k=1}^{\infty}\frac{k^4}{7k^5}=\sum_{k=1}^{\infty}\frac{1}{7k}=\frac{1}{7}\sum_{k=1}^{\infty}\frac{1}{k}$.

We will use the series $\frac{1}{7}\sum_{k=1}^{\infty}\frac{1}{k}$ to apply the Limit Comparison Test.
First, find the limit of the ratio of the terms of the two series:

$\lim_{k\to\infty}\frac{u_k}{v_k}=\lim_{k\to\infty}\frac{\frac{k^4+6k^3-1}{7k^5+k^2}}{\frac{1}{7k}}=\lim_{k\to\infty}\frac{7k^4+42k^3-7}{7k^4+k}=1$
Since $\lim_{k\to\infty}\frac{u_k}{v_k}=1$, by the Limit Comparison Test,
both $\sum_{k=1}^{\infty}\frac{k^4+6k^3-1}{7k^5+k^2}$ and $\frac{1}{7}\sum_{k=1}^{\infty}\frac{1}{k}$ either both converge or diverge.
But, $\frac{1}{7}\sum_{k=1}^{\infty}\frac{1}{k}$ is a harmonic series, which is a series that diverges.

Therefore, $\sum_{k=1}^{\infty}\frac{k^4+6k^3-1}{7k^5+k^2}$ diverges.

A simpler form of the Limit Comparison test, called the **Simplified Limit Comparison Test** , is as follows:

Suppose $\sum_{k=1}^{\infty} u_k$ and $\sum_{k=1}^{\infty} v_k$ are series with $u_k>0$ and $v_k>0$ for all k, and suppose $\lim_{k\to\infty}\frac{u_k}{v_k}=L>0$,
then either:

1. $\sum_{k=1}^{\infty} u_k$ and $\sum_{k=1}^{\infty} v_k$ both converge, or

2. $\sum_{k=1}^{\infty} u_k$ and $\sum_{k=1}^{\infty} v_k$ both diverge.

Let's apply the Simplified Limit Comparison Test and determine

if $\sum_{k=1}^{\infty} \frac{2}{8^k+5}$ converges or diverges.

The series $\sum_{k=1}^{\infty} \frac{2}{8^k+5}$ is a series without negative terms. We can apply the Simplified Limit Comparison Test by comparing the series $\sum_{k=1}^{\infty} \frac{2}{8^k+5}$ with the series $\sum_{k=1}^{\infty} \frac{2}{8^k}$ which is a convergent geometric series.

Then $\lim_{k\to\infty} \frac{\frac{2}{8^k+5}}{\frac{2}{8^k}} = \lim_{k\to\infty} \frac{8^k}{8^k+5} = 1 > 0$.

Thus, since $\sum_{k=1}^{\infty} \frac{2}{8^k}$ converges, then $\sum_{k=1}^{\infty} \frac{2}{8^k+5}$ also converges.

comparison tests

As we begin to compile a list of convergent and divergent series, new ones can sometimes be analyzed by comparing them to ones that we already understand.

Example 11.5.1 Does $\sum_{n=2}^{\infty} \frac{1}{n^2 \ln n}$ converge?

The obvious first approach, based on what we know, is the integral test. Unfortunately, we can't compute the required antiderivative. But looking at the series, it would appear that it must converge, because the terms we are adding are smaller than the terms of a p-series, that is,

$$\frac{1}{n^2 \ln n} < \frac{1}{n^2},$$

when $n \ge 3$. Since adding up the terms $1/n^2$ doesn't get "too big", the new series "should" also converge. Let's make this more precise.

The series $\sum_{n=2}^{\infty} \frac{1}{n^2 \ln n}$ converges if and only if $\sum_{n=3}^{\infty} \frac{1}{n^2 \ln n}$ converges—all we've done is dropped the initial term. We know that $\sum_{n=3}^{\infty} \frac{1}{n^2}$ converges. Looking at two typical partial sums:

$$s_n = \frac{1}{3^2 \ln 3} + \frac{1}{4^2 \ln 4} + \frac{1}{5^2 \ln 5} + \cdots + \frac{1}{n^2 \ln n} < \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots + \frac{1}{n^2} = t_n.$$

Since the p-series converges, say to L, and since the terms are positive, $t_n < L$. Since the terms of the new series are positive, the s_n form an increasing sequence and $s_n < t_n < L$ for all n. Hence the sequence $\{s_n\}$ is bounded and so converges.

Sometimes, even when the integral test applies, comparison to a known series is easier, so it's generally a good idea to think about doing a comparison before doing the integral test.

Example 11.5.2 Does $\sum_{n=1}^{\infty}\frac{|\sin n|}{n^2}$ converge?

We can't apply the integral test here, because the terms of this series are not decreasing. Just as in the previous example, however,

$$\frac{|\sin n|}{n^2}\le\frac{1}{n^2},$$

because $|\sin n|\le 1$. Once again the partial sums are non-decreasing and bounded above by $\sum 1/n^2=L$, so the new series converges.

Like the integral test, the comparison test can be used to show both convergence and divergence. In the case of the integral test, a single calculation will confirm whichever is the case. To use the comparison test we must first have a good idea as to convergence or divergence and pick the sequence for comparison accordingly.

Example 11.5.3 Does $\sum_{n=2}^{\infty}\frac{1}{\sqrt{n^2-3}}$ converge?

We observe that the -3 should have little effect compared to the n^2 inside the square root, and therefore guess that the terms are enough like $1/\sqrt{n^2}=1/n$ that the series should diverge. We attempt to show this by comparison to the harmonic series. We note that

$$\frac{1}{\sqrt{n^2-3}}>\frac{1}{\sqrt{n^2}}=\frac{1}{n},$$

so that

$$s_n=\frac{1}{\sqrt{2^2-3}}+\frac{1}{\sqrt{3^2-3}}+\cdots+\frac{1}{\sqrt{n^2-3}}>\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}=t_n,$$

where t_n is 1 less than the corresponding partial sum of the harmonic series (because we start at $n=2$ instead of $n=1$).
Since $\lim_{n\to\infty}t_n=\infty$, $\lim_{n\to\infty}s_n=\infty$ as well.

So the general approach is this: If you believe that a new series is convergent, attempt to find a convergent series whose terms are larger than the terms of the new series; if you believe that a new series is divergent, attempt to find a divergent series whose terms are smaller than the terms of the new series.

Example 11.5.4 Does $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n^2+3}}$ converge?

Just as in the last example, we guess that this is very much like the harmonic series and so diverges. Unfortunately, 1n2+3−−−−−√<1n,1n2+3<1n,

so we can't compare the series directly to the harmonic series. A little thought leads us to

1n2+3−−−−−√>1n2+3n2−−−−−−−√=12n,1n2+3>1n2+3n2=12n,

so if ∑1/(2n)∑1/(2n) diverges then the given series diverges. But since ∑1/(2n)=(1/2)∑1/n∑1/(2n)=(1/2)∑1/n, theorem 11.2.2 implies that it does indeed diverge.

For reference we summarize the comparison test in a theorem.

Theorem 11.5.5 *Suppose that* anan *and* bnbn *are non-negative for all* nn *and that* an≤bnan≤bn *when* n≥Nn≥N*, for some* NN*.*

If ∑n=0∞bn∑n=0∞bn *converges, so does* ∑n=0∞an∑n=0∞an*.*

If ∑n=0∞an∑n=0∞an *diverges, so does* ∑n=0∞bn∑n=0∞bn*.*

Cauchy's n^{th} root test

If you know that a series converges, then you can work further on it. But if it doesn't converge, then you can stop working on the series because you won't find an end to it. So how can you tell? Well, there's a test you can run.

The **root test** is a simple test that tests for absolute convergence of a series, meaning the series definitely converges to some value. This test doesn't tell you what the series converges to, just that your series converges.

The formal statement for the root test is:

For a series made up of terms a_n, define the limit as follows in this equation:

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} |\, a_n|^{\frac{1}{n}}$$

We then keep the following in mind:

- If L < 1, then the series absolutely converges.
- If L > 1, then the series diverges.
- If L = 1, then the series is either divergent or convergent.

That last statement basically means that if you get 1 for your L then your answer is unknown. The root test can't tell whether your series converges or diverges.

Now, let's take a look at using the root test for a converging series, a diverging series, and an unknown or indeterminate series.

Converging Series

First, let's look at a converging series. Here's the problem:

Use the root test to determine whether this series converges or diverges.

$$\sum \left(\frac{4n^4 + 58}{5n^4 - 3n^3} \right)^n$$

To use the root test, you'll follow the statement for the root test and take the limit of the absolute value of the terms in the series taken to the 1 / n power like this series of equations appearing here:

$$\sum \left(\frac{4n^4 + 58}{5n^4 - 3n^3} \right)^n$$

$$L = \lim_{n \to \infty} \sqrt[n]{|a_n|} = \lim_{n \to \infty} |\, a_n|^{\frac{1}{n}}$$

$$L = \lim_{n \to \infty} \sqrt[n]{\left| \frac{4n^4 + 58}{5n^4 - 3n^3} \right|^n} = \lim_{n \to \infty} \left| \frac{4n^4 + 58}{5n^4 - 3n^3} \right|$$

$$= \left| \frac{4}{5} \right|$$

$$= \frac{4}{5}$$

$$\frac{4}{5} < 1$$

D' Alemberts ratio test

In this article we will formulate the D' Alembert's Ratio Test on convergence of a series.

Let's start.

Statement of D'Alembert Ratio Test

A series ∑un∑un of positive terms is convergent if from and after some fixed term un+1un<r<1un+1un<r<1 , where r is a fixed number. The series is divergent if un+1un>1un+1un>1 from and after some fixed term.

D'Alembert's Test is also known as the **ratio test of convergence of a series**.

Theorem
Let ∑n=1∞an∑n=1∞an be a series of real numbers in RR, or a series of complex numbers in CC.

Let the sequence a_n satisfy:
$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = l$

- If $l > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.
- If $l < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.

Definitions for Generally Interested Readers

(Definition 1)

An infinite series $\sum u_n$ i.e. $u_1+u_2+u_3+\ldots+u_n$ is said to be convergent if S_n, the sum of its first n terms, tends to a finite limit S as n tends to infinity.

We call S the sum of the series, and write $S=\lim_{n\to\infty} S_n$.
Thus an infinite series $\sum u_n$ converges to a sum S, if for any given positive number ϵ, however small, there exists a positive integer n_0 such that $|S_n - S| < \epsilon$ for all $n \geq n_0$.

(Definition 2)

If $S_n \to \pm\infty$ as $n \to \infty$, the series is said to be **divergent**.
Thus, $\sum u_n$ is said to be divergent if for every given positive number λ, however large, there exists a positive integer n_0 such that $|S_n| > \lambda$ for all $n \geq n_0$.

(Definition 3)

If S_n does not tend to a finite limit, or to plus or minus infinity, the series is called **oscillatory**.

Proof & Discussions on Ratio Test

Let a series be $u_1+u_2+u_3+\ldots\ldots.$ We assume that the above inequalities are true.

- From the first part of the statement:
  $\frac{u_2}{u_1} < r$, $\frac{u_3}{u_2} < r$ ……… where r <1.
  Therefore $u_1+u_2+u_3+\ldots = u_1\left(1+\frac{u_2}{u_1}+\frac{u_3}{u_1}+\ldots\right)$
  $= u_1\left(1+\frac{u_2}{u_1}+\frac{u_3}{u_2}\times\frac{u_2}{u_1}+\ldots\right)$
  $< u_1(1+r+r^2+\ldots..)$
  Therefore, $\sum u_n < u_1(1+r+r^2+\ldots..)$
  or, $\sum u_n < \lim_{n\to\infty} \frac{u_1(1-r^n)}{1-r}$
  Since $r<1$, therefore as $n\to\infty$, $r^n \to 0$
  therefore $\sum u_n < \frac{u_1}{1-r}$ =k say, where k is a fixed number.
  Therefore $\sum u_n$ is convergent.

- Since, $\frac{u_{n+1}}{u_n}>1$ then, $\frac{u_2}{u_1}>1$, $\frac{u_3}{u_2}>1$ …….
  Therefore $u_2>u_1$, $u_3>u_2>u_1$, $u_4>u_3>u_2>u_1$ and so on.

  Therefore $\sum u_n=u_1+u_2+u_3+\ldots.+u_n$ > nu_1 . By taking n sufficiently large, we see that nu_1 can be made greater than any fixed quantity. Hence the series is divergent.

Academic Proof

From the statement of the theorem, it is necessary that $\forall n: a_n \ne 0$;
otherwise $\frac{a_{n+1}}{a_n}$ is not defined.
Here, $\left|\frac{a_{n+1}}{a_n}\right|$ denotes either the absolute value of $\frac{a_{n+1}}{a_n}$, or the complex modulus of $\frac{a_{n+1}}{a_n}$.

Absolute Convergence

Suppose $l<1$.
Let us take $\epsilon>0$ such that $l+\epsilon<1$.
Then:
$\exists N: \forall n>N: \left|\frac{a_n}{a_{n-1}}\right|<l+\epsilon$
Thus:
$|a_n| = \left|\frac{a_n}{a_{n-1}}\frac{a_{n-1}}{a_{n-2}}\cdots\frac{a_{N+2}}{a_{N+1}}a_{N+1}\right|$
$< (l+\epsilon)^{n-N-1}|a_{N+1}|$
By Sum of Infinite Geometric Progression, $\sum_{n=1}^{\infty}(l+\epsilon)^n$ converges.
So by the the corollary to the comparison test, it follows that $\sum_{n=1}^{\infty}|a_n|$ converges absolutely too.

Divergence

Suppose $l>1$.
Let us take $\epsilon>0$ small enough that $l-\epsilon>1$.
Then, for a sufficiently large N, we have:
$|a_n| = \left|\frac{a_n}{a_{n-1}}\frac{a_{n-1}}{a_{n-2}}\cdots\frac{a_{N+2}}{a_{N+1}}a_{N+1}\right|$
$> (l-\epsilon)^{n-N+1}|a_{N+1}|$
But $(l-\epsilon)^{n-N+1}|a_{N+1}| \to \infty$ as $n\to\infty$.
So $\sum_{n=1}^{\infty}a_n$ diverges.

Raabe's test

Raabe's test is the ratio test for convergence of a series.

Consider the limit $\lim_{n\to}\left|\frac{a_{n+1}}{a_n}\right| = L$

Raabe's test says that if $L<1$ then the series converges. If $L>1$ then the series diverges. If $L=1$ the test is inconclusive.

Proof:

We proceed by applying the limit comparsion test. This says that if the limit $\lim_{n\to\infty}\frac{|a_n|}{|b_n|}$ exists and is non-zero, then $\Sigma_1^\infty |a_n|$ converges if and only if $\Sigma_1^\infty |b_n|$ converges.

Note that if $|a_n|$ converges then a_n converges.

We compare a_n to the geomertic series r^m where

$$r = \frac{L+1}{2}$$

Now, if $L<1$ then $r>L$ and $\left|a(n+1)\right| < r|a_n|$ for $n>N$ (n large enough)

By induction

$|a_{n+m}| < r^m |a_n|$ for $n>N$ and $i>0$

$\Rightarrow$

because $r<1$

Therefore if $L<1$ the series a_n converges absolutely ($\Sigma_1^{\infty} a_n$ converges if $\Sigma_1^{\infty} |a_{N+m}|$ converges) .

If $L>1$ t hen $|a_{n+1}|>|a_n|$ for $n>N$. Such a series cannot converge, so the series diverges when $L>1$.

If $L=1$ we cannot show whether $\lim_{n\to\infty} \frac{|a_n|}{|b_n|}$ (where b_n is a geometric series) exists and is non-zero or not, so the test is inconclusive.

Unit - V

Alternating series and Maclaurin's series for sin x

In my previous post I said "recall the MacLaurin series for

$$\sin x:"$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

Since someone asked in a comment, I thought it was worth mentioning where this comes from. It would typically be covered in a second-semester calculus class, but it's possible to understand the idea with only a very basic knowledge of derivatives.

First, recall the derivatives $\sin'(x) = \cos(x)$ and $\cos'(x) = -\sin(x)$. Continuing, this means that the third derivative of $\sin(x)$ is $-\cos(x)$, and the derivative of that is $\sin(x)$ again. So the derivatives of $\sin(x)$ repeat in a cycle of length 4.

Now, suppose that an infinite series representation for $\sin(x)$ exists (it's not at all clear, a priori, that it should, but we'll come back to that). That is, something of the form

$$\sin(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

What could this possibly look like? We can use what we know about $\sin(x)$ and its derivatives to figure out that there is only one possible infinite series that could work.

First of all, we know that $\sin(0) = 0$. When we plug $x = 0$ into the above infinite series, all the terms with x in them cancel out, leaving only a_0: so a_0 must be 0.

Now if we take the first derivative of the supposed infinite series for $\sin(x)$, we get

$$a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \dots$$

We know the derivative of $\sin(x)$ is $\cos(x)$, and $\cos(0) = 1$: hence, using similar reasoning as before, we must have $a_1 = 1$. So far, we have

$$\sin(x) = x + a_2x^2 + a_3x^3 + \dots$$

Now, the second derivative of $\sin(x)$ is $-\sin(x)$. If we take the second derivative of this supposed series for $\sin(x)$, we get

$$2a_2 + (3 \cdot 2)a_3x + (4 \cdot 3)a_4x^2 + \dots$$

Again, since this should be $-\sin(x)$, if we substitute $x = 0$ we ought to get zero, so a_2 must be zero.

Taking the derivative a third time yields

$$(3 \cdot 2)a_3 + (4 \cdot 3 \cdot 2)a_4x + (5 \cdot 4 \cdot 3)a_5x^2 + \dots$$

and this is supposed to be $-\cos(x)$, so substituting $x = 0$ ought to give us -1: in order for that to happen we need $(3 \cdot 2)a_3 = -1$, and hence $a_3 = -1/6$.

To sum up, so far we have discovered that

$$\sin(x) = x - \frac{x^3}{6} + a_4x^4 + a_5x^5 + \dots$$

Do you see the pattern? When we take the nth derivative, the constant term is going to end up being $n! \cdot a_n$ (because it started out as a_nx^n and then went through n successive derivative operations before the x term disappeared: $a_n x^n \to n a_n x^{n-1} \to (n \cdot (n-1)) a_n x^{n-2} \to \cdots \to n! \cdot a_n$). If n is even, the nth derivative will be $\pm \sin(x)$, and so the constant term should be zero; hence all the even coefficients will be zero. If n is odd, the nth derivative will be $\pm \cos(x)$, and so the constant term should be ± 1: hence $n! \cdot a_n = \pm 1$, so $a_n = \pm 1/n!$, with the signs alternating back and forth. And this produces exactly what I claimed to be the expansion for $\sin x$:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

Using some other techniques from calculus, we can prove that this infinite series does in fact converge to $\sin x$, so even though we started with the potentially bogus assumption that such a series exists, once we have found it we can prove that it is in fact a valid representation of $\sin x$. It turns out that this same process can be performed to turn almost any function into an infinite series, which is called the Taylor series for the function (a MacLaurin series is a special case of a Taylor series). For example, you might like to try figuring out the Taylor series for $\cos x$, or for e^x (using the fact that e^x is its own derivative).

cos x

The Maclaurin series expansion for cos(x) is the infinite alternating series

=

Write a program in C using one or more iteration structures to approximate the cosine function, given a radian value x, input using scanf().
During each iteration of the summation shown above, use to output the value of the iteration index k, the kth approximation of cos(x), and the kth approximation error

where cosk(x) refers to the kth iteration approximation and cosstdlib(x) is the value returned by the C standard library cosine function.

To use the standard library cosine function, you must include the math header file:
#include <math.h>
Recall the factorial of a non negative integer n is defined as

0! = 1
You can implement the factorial evaluation using a while() construct. The pow() function calculates x^y
double pow(double x, double y) ;
The fab() function calculates the absoulute value of floating-point number,

double fabs(double x);
Example output:

Modify your program so you store the error computed during the previous iteration and stop iterating when the previous error equals the current iteration error. In your comment block header explain how many iterations were invoked for x = π, π/2, π/3, and π/4, for a reasonable approximate value of π.

log (1+x)

3. [7pts] The Taylor series for $f(x) = log(1+x)$ is

$$ln(x+1) = \sum_{k=1}^{n} (-1)^{k-1} \frac{x^k}{k} + E_n(\zeta, x) = P_n(x) + E_n(\zeta, x)$$

and converges for $x \in (-1, 1]$.

a) Use the Alternating Series Test to bound the error $|E_n|$ by $\hat{E}_n$. Use $\hat{E}_n$ to find an n sufficiently large so that $|ln(2) - P_n(2)| \leq 10^{-6}$.

b) One can accelerate the convergence rate using the following identity

$$ln(2) = ln(e \cdot 2/e) = 1 + log(2/e) = 1 + log(1+x)$$

Use the series above for this (new) x with $n \leq 10$ showing the accelerated convergence - a table of n versus the approximation and exact value $\ln 2 = 0.6931471806\ldots$.

(1+x)n

The Maclaurin series is the same as the Taylor series, except it is expanded around a=0.
So, you can start by assuming the Taylor series definition:
N∑n=0f(n)(a)n!(x−a)n
and modifying it to get:
N∑n=0f(n)(0)n!xn
Now, we can take the nth derivative. Let's say we go to n=3 only, because I know this is going to get a bit ridiculous to do.
f(0)(x)=f(x)=(1−x2)−1=11−x2
f'(x)=−(1−x2)−2(−2x)=(2x)(1−x2)−2=2x(1−x2)2
f''(x)=(2x)(−2(1−x2)−3(−2x))+(1−x2)−2(2)
=8x2(1−x2)−3+2(1−x2)−2=8x2(1−x2)3+2(1−x2)2
f'''(x)=[(8x2)(−3(1−x2)−4(−2x))+(1−x2)−3(16x)]+[2·(−2(1−x2)−3(−2x))]
=[(48x3)(1−x2)−4+(1−x2)−3(16x)]+[8x(1−x2)−3]
=48x3(1−x2)4+16x(1−x2)3+8x(1−x2)3
=48x3(1−x2)4+24x(1−x2)3
So the Maclaurin series up to n=3 is:
3∑n=0f(n)(0)n!xn
=11−a20!(x−a)0+2a(1−a2)21!(x−a)1+8a2(1−a2)3+2(1−a2)22!(x−a)2+48a3(1−a2)4+24a(1−a2)33!(x−a)3
=11−(0)20!x0+2(0)(1−(0)2)21!x1+8(0)2(1−(0)2)3+2(1−(0)2)22!x2+48(0)3(1−(0)2)4+24(0)(1−(0)2)33!x3+...
=1+x2+x4+...
The odd terms just go away. How convenient!

Definition of Convergence and Divergence in Series

The n^{th} partial sum of the series $\sum_{n=1}^{\infty} a_n$ is given by $S_n = a_1 + a_2 + a_3 + ... + a_n$. If the sequence of these partial sums $\{S_n\}$ converges to L, then the sum of the series converges to L. If $\{S_n\}$ diverges, then the sum of the series diverges.

Operations on Convergent Series

If $\sum a_n = A$, and $\sum b_n = B$, then the following also converge as indicated:

$$\sum ca_n = cA$$

$$\sum (a_n + b_n) = A + B$$
$$\sum (a_n - b_n) = A - B$$

Alphabetical Listing of Convergence Tests

Absolute Convergence

If the series $\sum_{n=1}^{\infty} |a_n|$ converges, then the series $\sum_{n=1}^{\infty} a_n$ also converges.

Alternating Series Test

If for all n, a_n is positive, non-increasing (i.e. $0 < a_{n+1} <= a_n$), and approaching zero, then the alternating series

$$\sum_{n=1}^{\infty} (-1)^n a_n \quad \text{and} \quad \sum_{n=1}^{\infty} (-1)^{n-1} a_n$$

both converge.
If the alternating series converges, then the remainder $R_N = S - S_N$ (where S is the exact sum of the infinite series and S_N is the sum of the first N terms of the series) is bounded by $|R_N| <= a_{N+1}$

Deleting the first N Terms

If N is a positive integer, then the series

$$\sum_{n=1}^{\infty} a_n \text{ and } \sum_{n=N+1}^{\infty} a_n$$

both converge or both diverge.

Direct Comparison Test

If $0 <= a_n <= b_n$ for all n greater than some positive integer N, then the following rules apply:

If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

Geometric Series Convergence

The geometric series is given by

$\sum_{n=0}^{\infty} a\, r^n = a + a\, r + a\, r^2 + a\, r^3 + ...$

If $|r| < 1$ then the following geometric series converges to $a / (1 - r)$.

If $|r| >= 1$ then the above geometric series diverges.

Integral Test

If for all $n >= 1$, $f(n) = a_n$, and f is positive, continuous, and decreasing then

$\sum_{n=1}^{\infty} a_n$ and $\int_1^{\infty} a_n$

either both converge or both diverge.

If the above series converges, then the remainder $R_N = S - S_N$ (where S is the exact sum of the infinite series and S_N is the sum of the first N terms of the series) is bounded by $0 <= R_N <= \int (N..\infty)\, f(x)\, dx$.

Limit Comparison Test

If lim $(n \to \infty)$ $(a_n / b_n) = L$,
where $a_n, b_n > 0$ and L is finite and positive,
then the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ either both converge or both diverge.

n^{th}-Term Test for Divergence

If the sequence $\{a_n\}$ does not converge to zero, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

p-Series Convergence

The p-series is given by

$$\sum_{n=1}^{\infty} 1/n^p = 1/1^p + 1/2^p + 1/3^p + ...$$

where $p > 0$ by definition.
If $p > 1$, then the series converges.
If $0 < p <= 1$ then the series diverges.

Ratio Test

If for all n, $n \neq 0$, then the following rules apply:
Let $L = \lim (n \to \infty) | a_{n+1} / a_n |$.

If $L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges.

If $L > 1$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

If $L = 1$, then the test in *inconclusive*.

Root Test

Let $L = \lim (n \to \infty) | a_n |^{1/n}$.

If $L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges.

If $L > 1$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

If $L = 1$, then the test in *inconclusive*.

Taylor Series Convergence

If f has derivatives of all orders in an interval I centered at c, then the Taylor series converges as indicated:

$$\sum_{n=0}^{\infty} (1/n!)\, f^{(n)}(c)\, (x - c)^n = f(x)$$

if and only if $\lim (n \to \infty) RN = 0$ for all x in I.

The remainder $R_N = S - S_N$ of the Taylor series (where S is the exact sum of the infinite series and S_N is the sum of the first N terms of the series) is equal to $(1/(n+1)!)\ f^{(n+1)}(z)\ (x - c)^{n+1}$, where z is some constant between x and c.

Applications of mean value theorem to monotonic functions and inequalities

Well, the MVT tells us that for any interval [a,b], we can find an interior point c∈(a,b) where the derivative/"sensitivity" f′(c) "represents" the overall change of the function, in the sense that $\Delta f=f(b)-f(a)$ equals $f'(c)\Delta x=f'(c)(b-a)$.

So the rough answer is that if we have information about the derivative/"sensitivity" of a function at lots of points (e.g. on an interval [a,b]), then we can use the MVT to translate that into properties of the function itself. For example, in (1) below, we're given lots of "local" derivative information, e.g. that $f'(x)>0$ always (the derivative is always positive), and we can use MVT to translate that into a more "global" statement about increasing functions (i.e. $f(a)<f(b)$ whenever $a<b$).

* Well, the rough answer is that it allows us to use **"local" derivative/"sensitivity"** information to say interesting things about **"global"/big-picture function** properties. (The limit definition of a derivative already tells us how to use **"global"/big-picture function** information---namely the function values themselves---to say interesting things about **"local" derivative/"sensitivity"** properties.) In applications 1, 2, and 3 below, this is how the MVT is used. *

On the other hand, the MVT can seem kind of arbitrary (&), especially if you don't care so much about rigor. Is it really necessary to learn/teach?

Well, you can read the thread if you're interested. But for me, here's the key takeaway, stolen from Jeff Strom's excellent answer:

The MVT is the basis for all proofs that geometric intuition about slopes of tangent lines holds for the limit definition. That is, the metamathematical content of the MVT is that the intuition definition matches the formal definition.

In other words, the MVT is just a rigorous way to capture our geometric intuition about derivatives, which you use a lot in sketching derivatives. But again, don't worry if you find the MVT arbitrary. You'll be fine if you understand the intuition for some of its most important applications:

(1) Increasing and decreasing (monotone) functions in terms of derivatives.

(2) Functions with always-zero derivatives (this is later applied to answer the very important question of whether anti-derivatives are unique, but don't worry if you haven't heard of anti-derivatives yet).

(3) When tangents lie above or below graphs.

In all of these, we have some intuition involving some sort of inequality (&&) in a function or graph: the notion of increasing/decreasing in (1), a graph "resembling a horizontal line everywhere" in (2), and the notion of above/below in (3). And in each case, the MVT is just a simple way to formalize our graphical intuition.

* (4) Precursor to Taylor's "Weak" Theorem *

Additional comments (feel free to ignore these)

(&) The reason MVT seems arbitrary is because it just says there exists some point c with $f'(c)=f(b)-f(a)b-a$, while it would possibly be more natural to have a statement like "the average value of the derivative is $f(b)-f(a)b-a$". And it turns out that this is true, in some sense (it's half of the fundamental theorem of calculus, which you'll encounter later on). The caveat? To prove that, we use (2) from above, which comes from MVT.

(&&) More precisely, it's worth clarifying why MVT works to prove these inequalities, because (&) makes MVT seem kind of unnatural or artificial. Roughly speaking, the reason MVT works for 1, 2, and 3 is that **the information on derivatives we have in these cases is uniform:**

(1) We're using MVT on a function with $f'(x)>0$ for all x.

(2) We're using MVT on a function with $f'(x)=0$ for all x.

(3) We're using MVT on a function with $f''(x)>0$ for all x.

Maxima and minima

As the name suggests, this topic is devoted to the method of finding the maximum and the minimum values of a function in a given domain. It finds application in almost every field of work, and in every subject. Let’s find out more about the maxima and minima in this topic.

Some day-to-day applications are described below:

- To an engineer – The maximum and the minimum values of a function can be used to determine its boundaries in real-life. For example, if you can find a suitable function for the speed of a train; then determining the maximum possible speed of the train can help you choose the materials that would be strong enough to withstand the pressure due to such high speeds, and can be used to manufacture the brakes and the rails etc. for the train to run smoothly.
- To an economist – The maximum and the minimum values of the total profit function can be used to get an idea of the limits the company must put on the salaries of the employees, so as to not go in loss.
- To a doctor – The maximum and the minimum values of the function describing the total thyroid level in the bloodstream can be used to determine the dosage the doctor needs to prescribe to different patients to bring their thyroid levels to normal.

Types of Maxima and Minima

The maxima or minima can also be called an extremum i.e. an extreme value of the function. Let us have a function y = f(x) defined on a known domain of x. Based on the interval of x, on which the function attains an extremum, the extremum can be termed as a 'local' or a 'global' extremum. Let's understand it better in the case of maxima.

Browse more Topics under Application Of Derivatives

- Rate of Change of Quantities
- Approximations
- Increasing and Decreasing Functions
- Tangents and Normals

Local Maxima

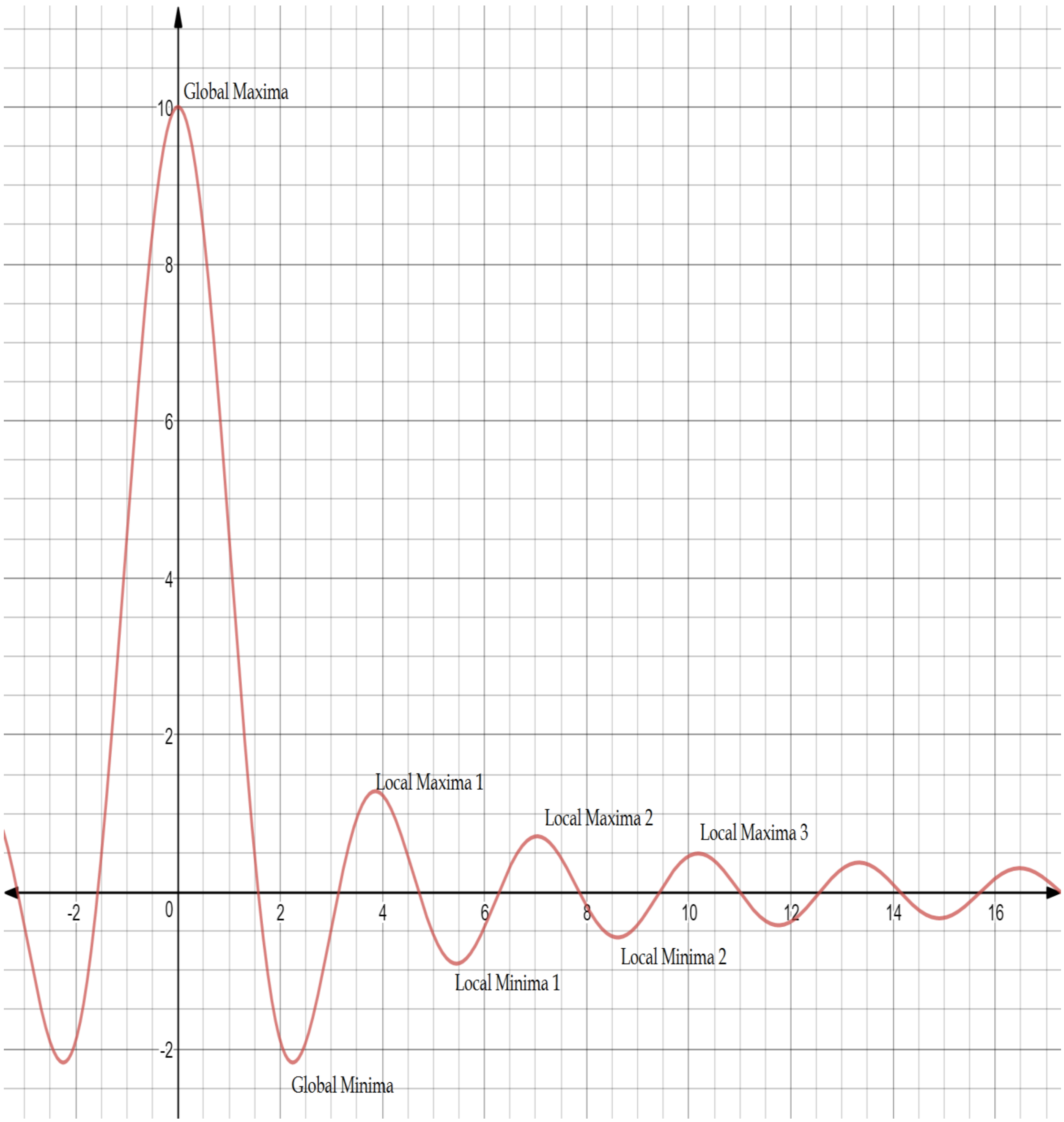
A point is known as a Local Maxima of a function when there may be some other point in the domain of the function for which the value of the function is more than the value of the local maxima, but such a point doesn't exist in the vicinity or neighborhood of the local maxima. You can also understand it as a maximum value with respect to the points nearby it.

Global Maxima

A point is known as a Global Maxima of a function when there is no other point in the domain of the function for which the value of the function is more than the value of the global maxima. Types of Global Maxima:

- Global maxima may satisfy all the conditions of local maxima. You can also understand it as the Local Maxima with the maximum value in this case.
- Alternately, the global maxima for an increasing function could be the endpoint in its domain; as it would obviously have the maximum value. In this case, it isn't a local maximum for the function.

Similarly, the local and the global minima can be defined. Look at the graph below to identify the different types of maxima and minima.



Stationary Points

A stationary point on a curve is defined as one at which the derivative vanishes i.e. a point (x_0, f(x_0)) is a stationary point of f(x) if [dfdx]x=x0=0[dfdx]x=x0=0. Types of stationary points:

- Local Maxima
- Local Minimas
- Inflection Points

We won't discuss inflection points here. As of now though, you must note that all the points of extremum are stationary points.

Proof: I'll prove the above statement for the case of a Local Maxima. Others will simply follow from this. Let us have a function y = f(x) that attains a Local Maximum at point x = x_0. Near the extremum point, the curve will look something like this:

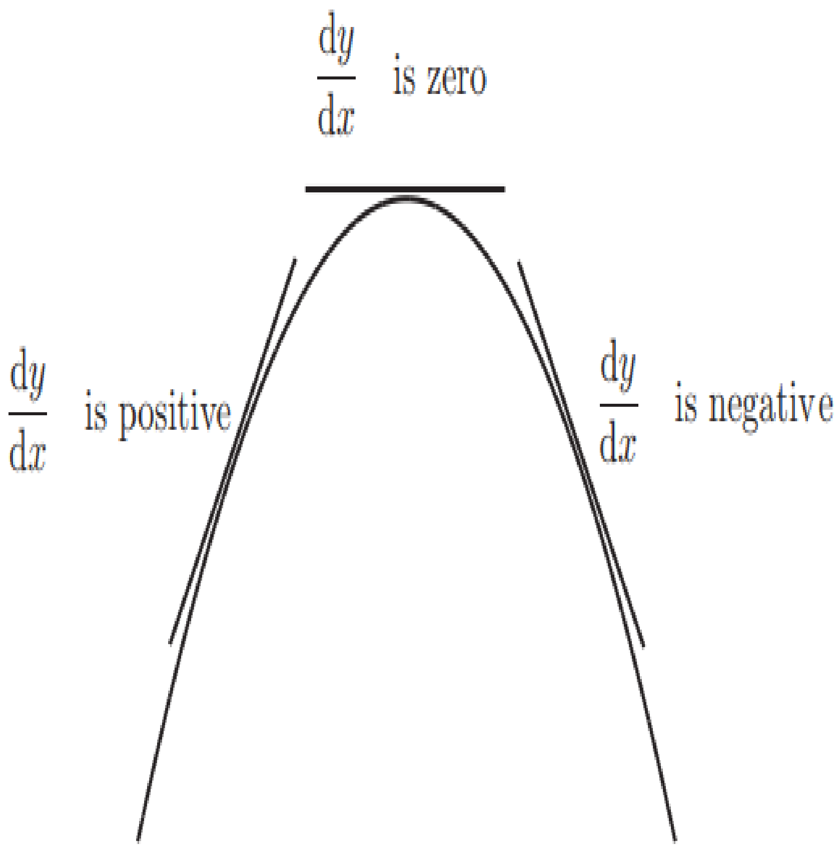


Fig 1.

Clearly, the derivative of the function has to go to 0 at the point of Local Maximum; otherwise, it would never attain a maximum value with respect to its neighbors.

The Second Derivative Test

This test is used to determine whether a stationary point is a Local Maxima or a Local Minima. Whether it is a global maxima/global minima can be determined by comparing its value with other local maxima/minima. Let us have a function y = f(x) with x = x_0 as a stationary point. Then the test says:

- If [d2fd2x]x=x0<0[d2fd2x]x=x0<0, then x = x_0 is a point of Local Maxima.
- If [d2fd2x]x=x0>0[d2fd2x]x=x0>0, then x = x_0 is a point of Local Minima.
- If [d2fd2x]x=x0=0[d2fd2x]x=x0=0, then check in the following way:
  - If for x > x_0, [dfdx]x=x0<0[dfdx]x=x0<0 and for x < x_0, [dfdx]x=x0>0[dfdx]x=x0>0 i.e. the function is increasing for x < x_0 and decreasing for x > x_0; we can conclude that x = x_0 is a point of Local Maxima.
  - Similarly, if for x > x_0, [dfdx]x=x0>0[dfdx]x=x0>0 and for x < x_0, [dfdx]x=x0<0[dfdx]x=x0<0 i.e. the function is decreasing for x < x_0 and increasing for x > x_0; we can conclude that x = x_0 is a point of Local Minima.

The proof of the third case can be understood by looking at Fig 1. above for local maxima. Similarly, for local minima, one can get:

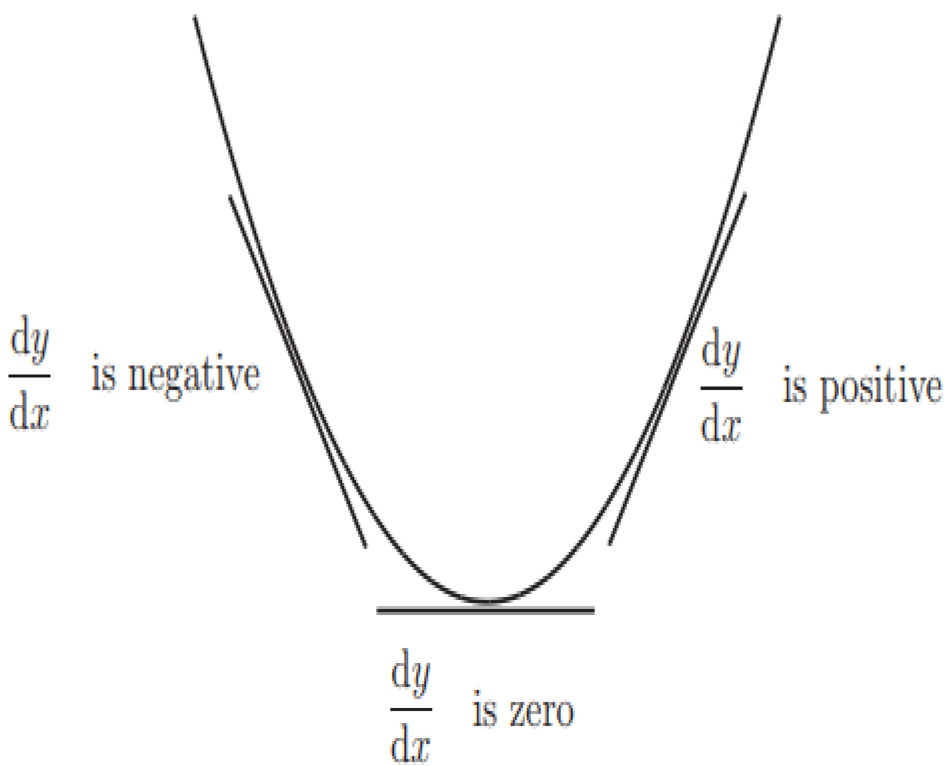


Fig 2.

Proof of the Second Derivative Test

We'll prove the test for the case of a Local Minima. The proof for a Local Maxima will follow in a similar fashion. Take a look at the Fig 2. above. One can see that the slope of the tangent drawn at any point on the curve i.e. dtdxdtdx changes from a negative value to 0 to a positive value, near the point of local minima. T
his means that the function that is represented by(say) f(x)=dtdxf(x)=dtdx behaves like an increasing function. The condition for a function to be increasing is:dfdx>0 i.e.d2yd2x>0dfdx>0 i.e.d2yd2x>0This confirms that the function will have a local minima if the first derivative is 0, and the second derivative is positive at that point.

Solved Examples for You on Maxima and Minima

Question 1 : Find the local maxima and minima for the function $y = x^3 – 3x + 2$.

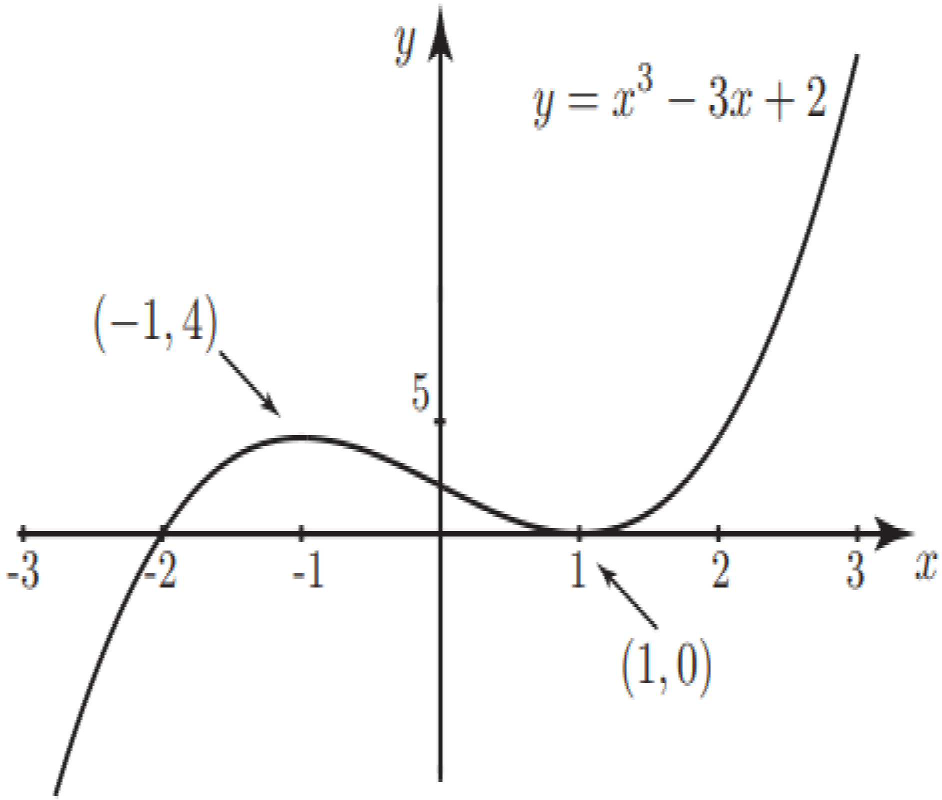
Answer : We'll need to find the stationary points for this function, for which we need to calculate dfdxdfdx. We'll proceed as follows:
y=x3–3x+2y=x3–3x+2
dydx=3x2–3dydx=3x2–3
At stationary points, dydx=0dydx=0. Thus, we have;
3x2–3=03x2–3=0
3(x2–1)=03(x2–1)=0
(x–1)(x+1)=0(x–1)(x+1)=0
x = 1 / x = -1 x = 1 / x = -1

Now we have to determine whether any of these stationary points are extremum points. We'll use the second derivative test for this:

dydx=3x2–3dydx=3x2–3
d2yd2x=6xd2yd2x=6x

- For x = 1; d2yd2x=6/times1=6d2yd2x=6/times1=6, which is positive. Thus the point (1, y(x = 1)) is a point of Local Minima.
- For x = -1; d2yd2x=6/times−1=−6d2yd2x=6/times−1=−6, which is positive. Thus the point (-1, y(x = -1)) is a point of Local Maxima.

We can see from the graph below to verify our calculations:



This concludes our discussion on this topic of maxima and minima.

Question 2: What are relative maxima and relative minima?

Answer: Finding out the relative maxima and minima for a function can be done by observing the graph of that function. A relative maxima is the greater point than the points directly beside it at both sides. Whereas, a relative minimum is any point which is lesser than the points directly beside it at both sides.

Question 3: How to find out the absolute maxima of a function?

Answer: Finding the absolute maxima:

Firstly, find out all the critical numbers of the function within the interval [a, b].

Then, plug in every single critical number from the first step into the function i.e. f(x).

Plugin the ending points that are (a) and (b) into the function f(x).

Finally, the biggest value is the absolute maxima and the lowest value is the absolute minima.

Question 4: What is the absolute maxima?

Answer: The biggest value that a mathematical function can consume over its whole curve. The absolute maxima on the graph takes place at x = d, and the absolute minima of that graph takes place at x = a.

Question 5: What are the local and global maxima and minima?

Answer: The global maxima and minima of any function are known as the global extrema of that function. Whereas, the local maxima and minima are said to be the local extrema.

Indeterminant forms (applications of Maxima and Minima to simple Problems)

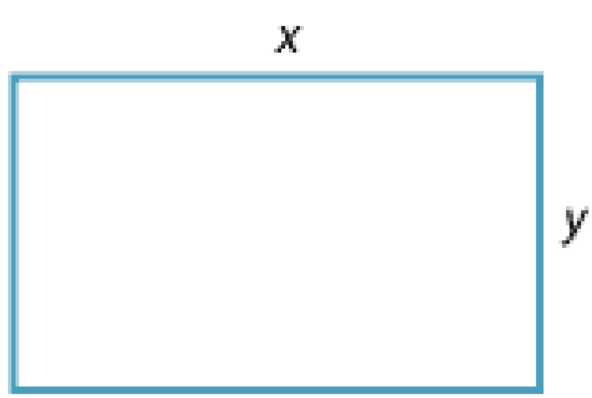
The need to find local maxima and minima arises in many situations. The first example we will look at is very familiar, and can also be solved without using calculus. Examples of solving such problems without the use of calculus can be found in the module *Quadratics* .

Example

Find the dimensions of a rectangle with perimeter 1000 metres so that the area of the rectangle is a maximum.

Solution

Let the length of the rectangle be xx m, the width be yy m, and the area be AA m22.



The perimeter of the rectangle is 1000 metres. So

$$1000=2x+2y,1000=2x+2y,$$

and hence

$$y=500-x.y=500-x.$$

The area is given by $A=xyA=xy$. Thus

$$A(x)=x(500-x)=500x-x2.(1)A(x)=x(500-x)=500x-x2.(1)$$

Because xx and yy are lengths, we must have $0\le x\le 5000\le x\le 500$.

The problem now reduces to finding the value of xx in $[0,500][0,500]$ for which AA is a maximum. Since AA is differentiable, the maximum must occur at an endpoint or a stationary point.

From (1), we have

$$dAdx=500-2x.dAdx=500-2x.$$

Setting $dAdx=0dAdx=0$ gives $x=250x=250$.

Hence, the possible values for AA to be a maximum are $x=0x=0$, $x=250x=250$ and $x=500x=500$. Since $A(0)=A(500)=0A(0)=A(500)=0$, the maximum value of AA occurs when $x=250x=250$.

The rectangle is a square with side lengths 250 metres. The maximum area is 62 50062 500 square metres.

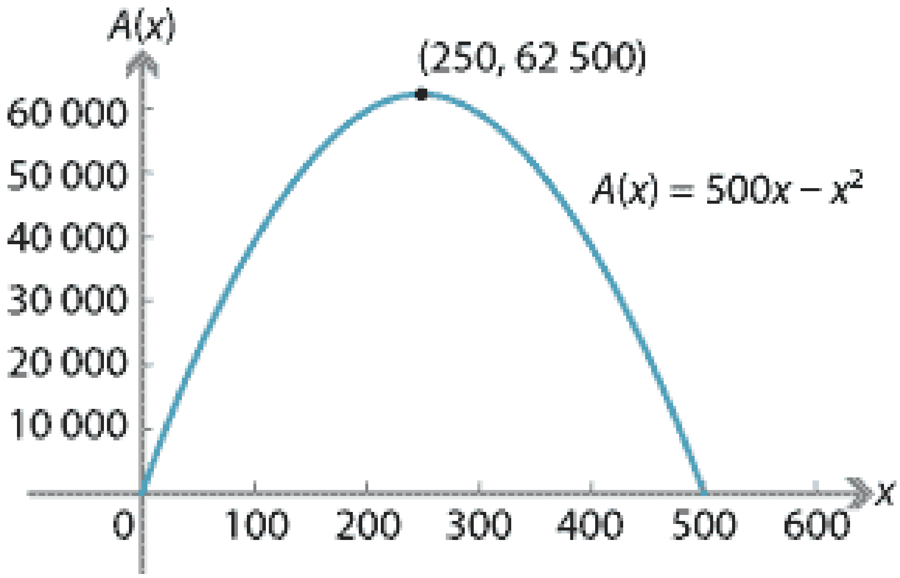
Notes.

1. $dAdx>0dAdx>0$, for $0\le x<2500\le x<250$, and $dAdx<0dAdx<0$, for $250<x\le 500250<x\le 500$. Hence, there is a local maximum at $x=250x=250$.

2. $d2Adx2=-2<0d2Adx2=-2<0$. This is a second way to see that $x=250x=250$ is a local maximum.

3. The graph of $A(x)=500x-x2A(x)=500x-x2$ is a parabola with a negative coefficient of $x2x2$ and a turning point at $x=250x=250$. This is a third way of establishing the local maximum.

4. It is worth looking at the graph of $A(x)A(x)$ against xx.



Exercise 9

A farmer has 8 km of fencing wire, and wishes to fence a rectangular piece of land. One boundary of the land is the bank of a straight river. What are the dimensions of the rectangle so that the area is maximised?

The following steps provide a general procedure which you can follow to solve maxima and minima problems.

Steps for solving maxima and minima problems

Step 1.

Where possible draw a diagram to illustrate the problem. Label the diagram and designate your variables and constants. Note any restrictions on the values of the variables.

Step 2.

Write an expression for the quantity that is going to be maximised or minimised. Eliminate some of the variables. Form an equation for this quantity in terms of a single independent variable. This may require some algebraic manipulation.

Step 3.

If $y=f(x)y=f(x)$ is the quantity to be maximised or minimised, find the values of xx for which $f'(x)=0f'(x)=0$.

Step 4.

Test each point for which $f'(x)=0f'(x)=0$ to determine if it is a local maximum, a local minimum or neither.

Step 5.

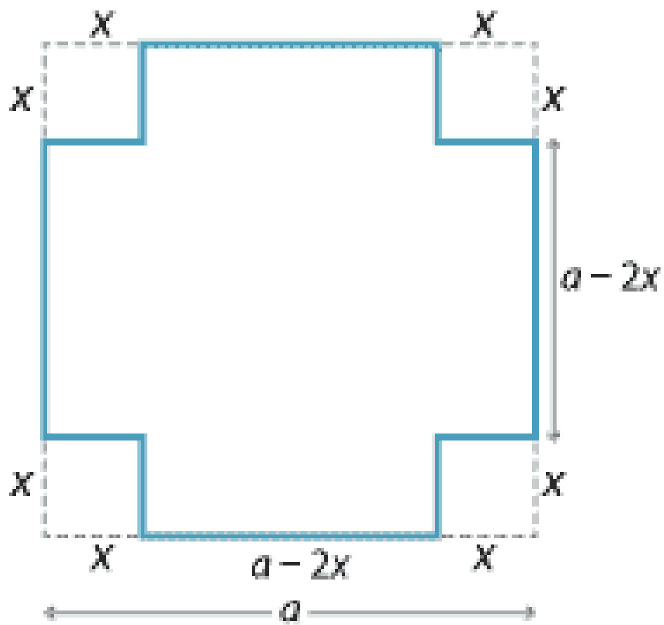
If the function y=f(x)y=f(x) is defined on an interval, such as [a,b][a,b] or [0,∞)[0,∞), check the values of the function at the end points.

Example

A square sheet of cardboard with each side aa centimetres is to be used to make an open-top box by cutting a small square of cardboard from each of the corners and bending up the sides. What is the side length of the small squares if the box is to have as large a volume as possible?

Solution

Step 1.



Let the side length of the small squares be xx cm. The side length of the open box is (a−2x)(a−2x) cm, and the height is xx cm. Here aa is a constant, and xx is the variable we will work with. We must have

0≤x≤a2.0≤x≤a2.

Step 2.

The volume VV cm33 of the box is given by

V(x)=x(a−2x)2=4x3−4ax2+a2x.V(x)=x(a−2x)2=4x3−4ax2+a2x.

Step 3.

We have

dVdx=12x2−8ax+a2=(2x−a)(6x−a).dVdx=12x2−8ax+a2=(2x−a)(6x−a).
Thus dVdx=0dVdx=0 implies x=a2x=a2 or x=a6x=a6.

Step 4.

We note that x=a2x=a2 is an endpoint and that V(a2)=0V(a2)=0. We will use the second derivative test for x=a6x=a6. We have

d2Vdx2=24x−8a=8(3x−a).d2Vdx2=24x−8a=8(3x−a).

When x=a6x=a6, we get

d2Vdx2=8(3×a6−a)=−4a<0.d2Vdx2=8(3×a6−a)=−4a<0.

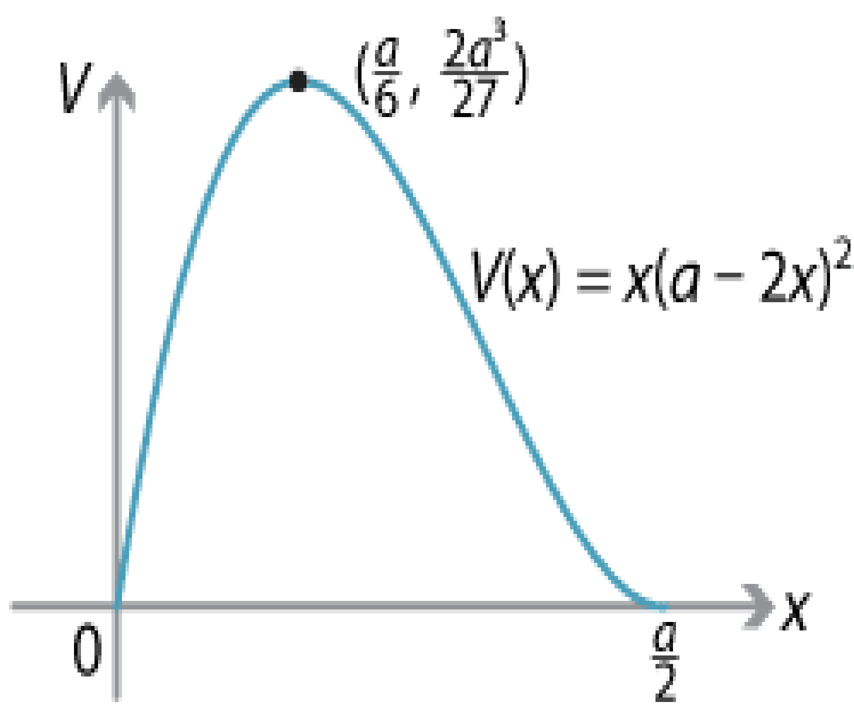
Hence, x=a6x=a6 is a local maximum.

Step 5.

The maximum value of the function is at x=a6x=a6,
as V(0)=V(a2)=0V(0)=V(a2)=0. The maximum volume is

V(a6)=2a327.V(a6)=2a327.

The following diagram shows the graph of VV against xx.

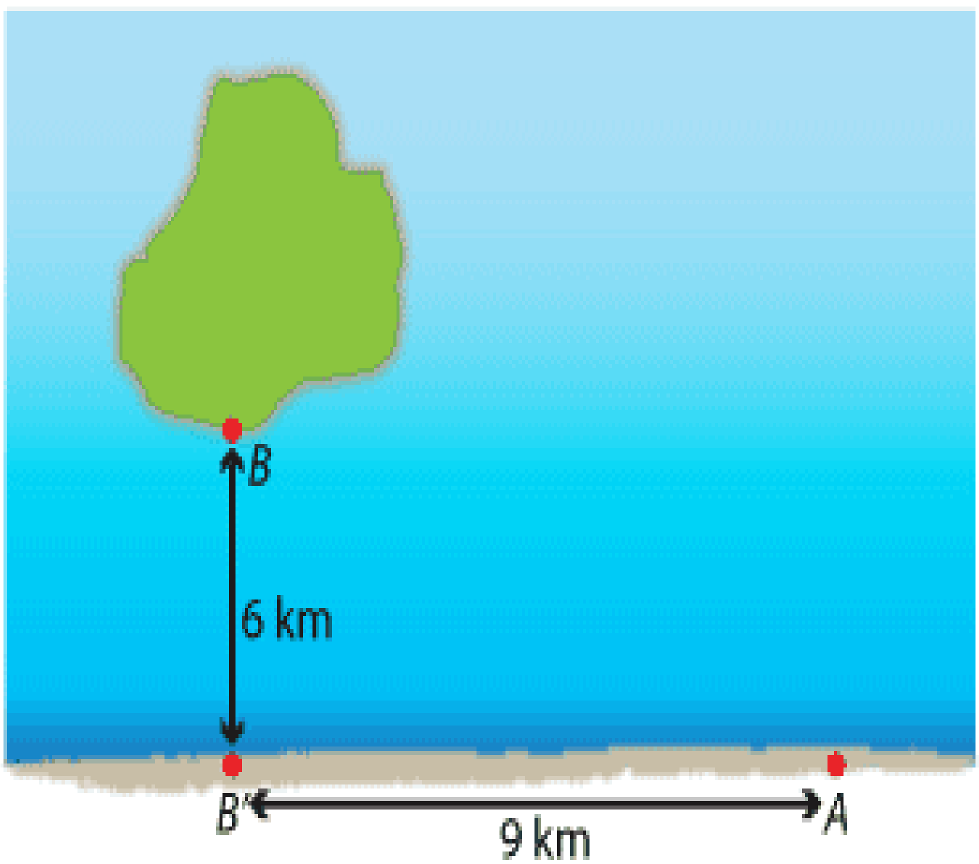


The following example illustrates a number of issues that can occur.

Example

A company wants to run a pipeline from a point AA on the shore to a point BB on an island which is 6 km from the shore. It costs PP per kilometre to run the pipeline on shore, and QQ per kilometre to run it underwater. There is a point B′B′ on the shore so that BB′BB′ is at right angles to AB′AB′. The straight shoreline is the line AB′AB′. The distance AB′AB′ is 9 km. Find how the pipeline should be laid to minimise the cost if

1. P=4000P=4000 and Q=5000Q=5000
2. P=5000P=5000 and Q=13 000Q=13 000
3. P=24 000P=24 000 and Q=25 000Q=25 000.

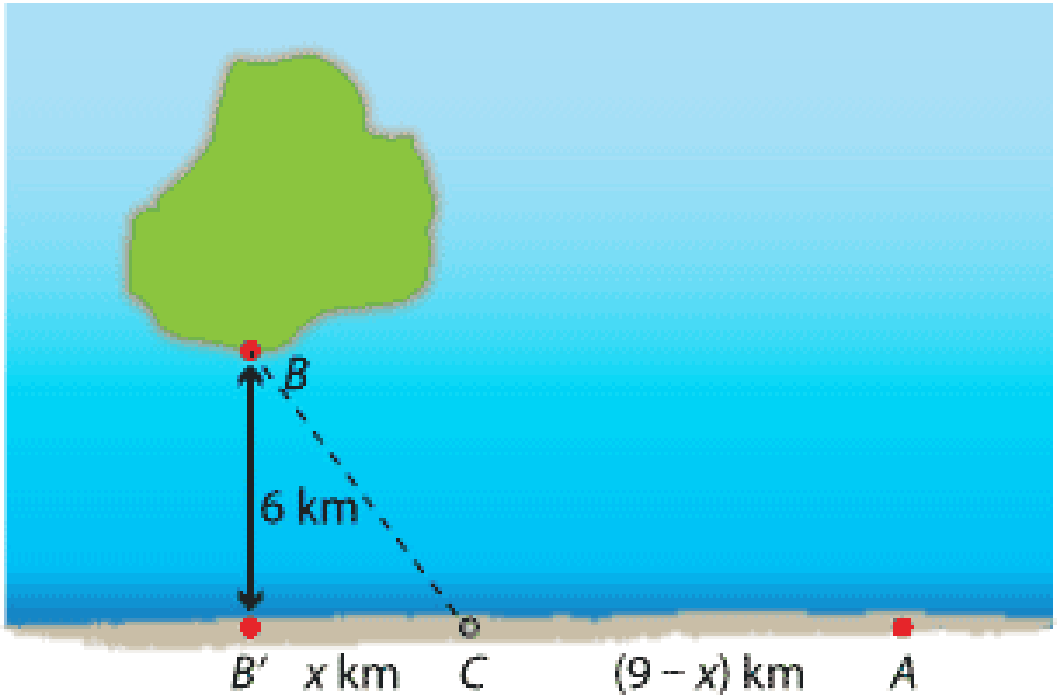


Detailed description

Solution

We will work through most of the problem without assigning values to PP and QQ.

Step 1.



Suppose that the pipeline leaves the shore xx km from B′B′ at a point CC between B′B′ and AA. The distance ACAC is (9−x)(9−x) km. By Pythagoras' theorem, the distance CBCB is 36+x2−−−−−−√36+x2 km. It is important to note that

0≤x≤9.0≤x≤9.

Step 2.

Let TT be the total cost. Then

T(x)=P(9−x)+Q36+x2−−−−−−√.(1)T(x)=P(9−x)+Q36+x2.(1)

Step 3.

We have

dTdx=Qx36+x2−−−−−−√−P.dTdx=Qx36+x2−P.

Hence, solving dTdx=0dTdx=0 gives

Qx36+x2−−−−−−√−PQxQ2x2(Q2−P2)x2x=0=P36+x2−−−−−−√=P2(36+x2)=36P2=36P2Q2−P2−−−−−−−−−√=6PQ2−P2−−−−−−−√.(2)Qx36+x2−P=0Qx=P36+x2Q2x2=P2(36+x2)(Q2−P2)x2=36P2x=36P2Q2−P2=6PQ2−P2.(2)

Note that we need Q>PQ>P for this solution xx to exist, and we also need 0≤x≤90≤x≤9. If Q≤PQ≤P, the pipeline should go directly from AA to BB, with minimum cost $313−−√Q$313Q.

Step 4.

Using the second derivative test:

d2Tdx2=36Q(36+x2)32>0,d2Tdx2=36Q(36+x2)32>0,

for all xx. Hence, there is a local minimum at x=6PQ2−P2−−−−−−−√x=6PQ2−P2 for the function with rule T(x)T(x). Such a local minimum may occur outside the interval [0,9][0,9].

Step 5.

- If x=0x=0, then T=9P+6QT=9P+6Q.
- If x=9x=9, then T=313−−√QT=313Q.
- If x=6PQ2−P2−−−−−−−√x=6PQ2−P2, then from (1) we have

T=P(9−6PQ2−P2−−−−−−−√)+Q36+36P2Q2−P2−−−−−−−−−−−−√=9P−6P2Q2−P2−−−−−−−√+Q36(Q2−P2)+36P2Q2−P2−−−−−−−−−−−−−−−−−√=9P−6P2Q2−P2−−−−−−−√+6Q2Q2−P2−−−−−−−√=9P+6Q2−P2−−−−−−−√.(3)T=P(9−6PQ2−P2)+Q36+36P2Q2−P2=9P−6P2Q2−P2+Q36(Q2−P2)+36P2Q2−P2=9P−6P2Q2−P2+6Q2Q2−P2=9P+6Q2−P2.(3)

The local minimum occurs in the interval [0,9][0,9] if and only if

6PQ2−P2−−−−−−−√≤9.6PQ2−P2≤9.

We now solve this inequality for the ratio PQPQ, assuming that Q>PQ>P:

6PQ2−P2−−−−−−−√≤9 ⟺36P2Q2−P2≤81⟺36P2≤81(Q2−P2)⟺117P2≤81Q2⟺P2Q2≤81117.6PQ2−P2≤9 ⟺36P2Q2−P2≤81⟺36P2≤81(Q2−P2)⟺117P2≤81Q2⟺P2Q2≤81117.

Thus the local minimum occurs in the interval [0,9][0,9] if and only if PQ≤313−−√PQ≤313.

We now consider the particular values of PP and QQ specified in the question.

1. P=4000P=4000 and Q=5000Q=5000. By equation (1), we have

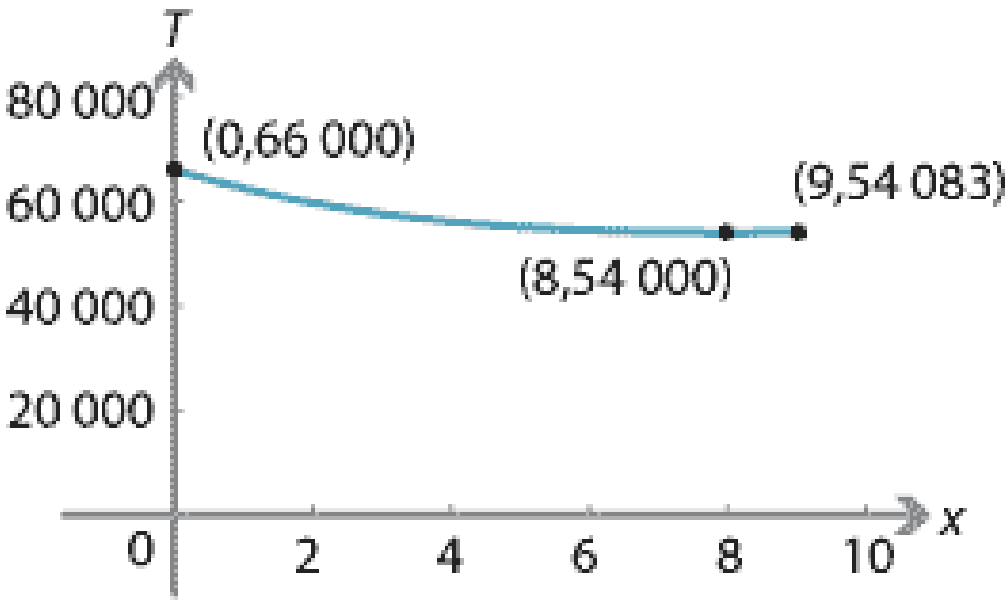
T=4000(9−x)+500036+x2−−−−−−√,for 0≤x≤9.T=4000(9−x)+500036+x2,for 0≤x≤9.

Note that

T(0)T(9)=36000+30000=66000=1500013−−√≈54083.T(0)=36000+30000=66000 T(9)=1500013≈54083.

By equation (2), the local minimum point is x=8x=8 and in this case, by equation (3), the minimum cost is

Tmin=9×4000+6×3000=\$54000.Tmin=9×4000+6×3000=\$54000.



2. P=5000P=5000 and Q=13000Q=13000. By equation (1), we have

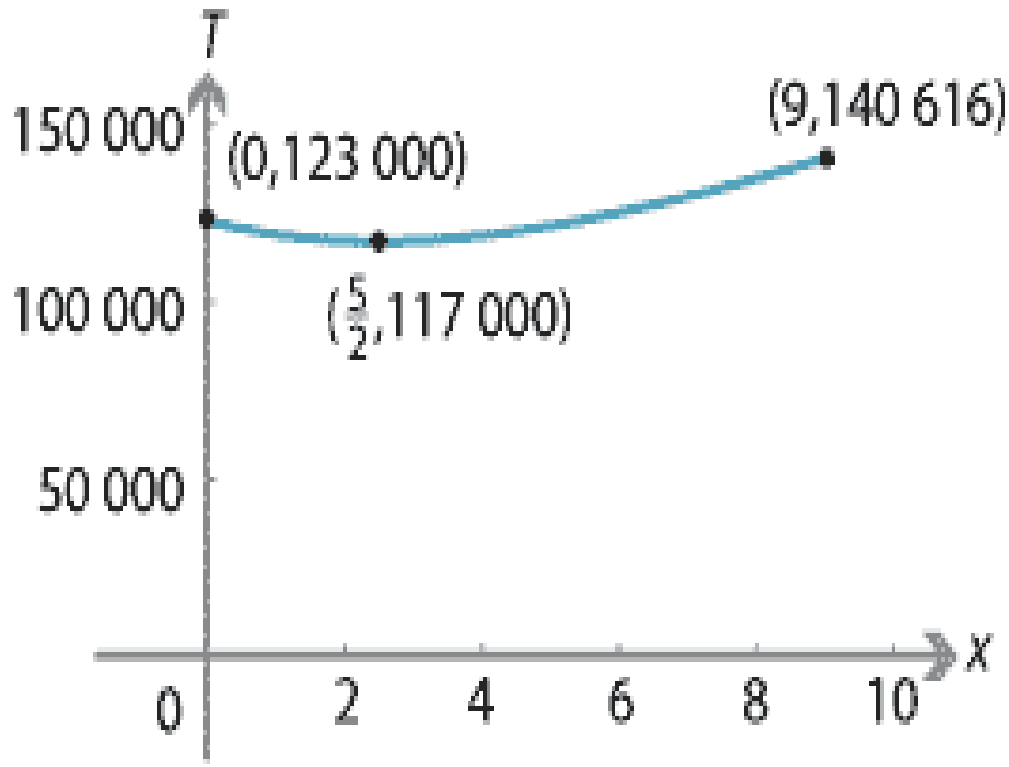
T=5000(9−x)+1300036+x2−−−−−−√,for 0≤x≤9.T=5000(9−x)+1300036+x2,for 0≤x ≤9.

We note that

T(0)=123000,T(9)=3900013−−√≈140616.T(0)=123000,T(9)=3900013≈140616.

By equation (2), the local minimum point is x=52x=52 and in this case, by equation (3), the minimum cost is

Tmin=9×5000+6×12000=\$117000.Tmin=9×5000+6×12000=\$117000.



3. P=24000P=24000 and Q=25000Q=25000. By equation (1), we have

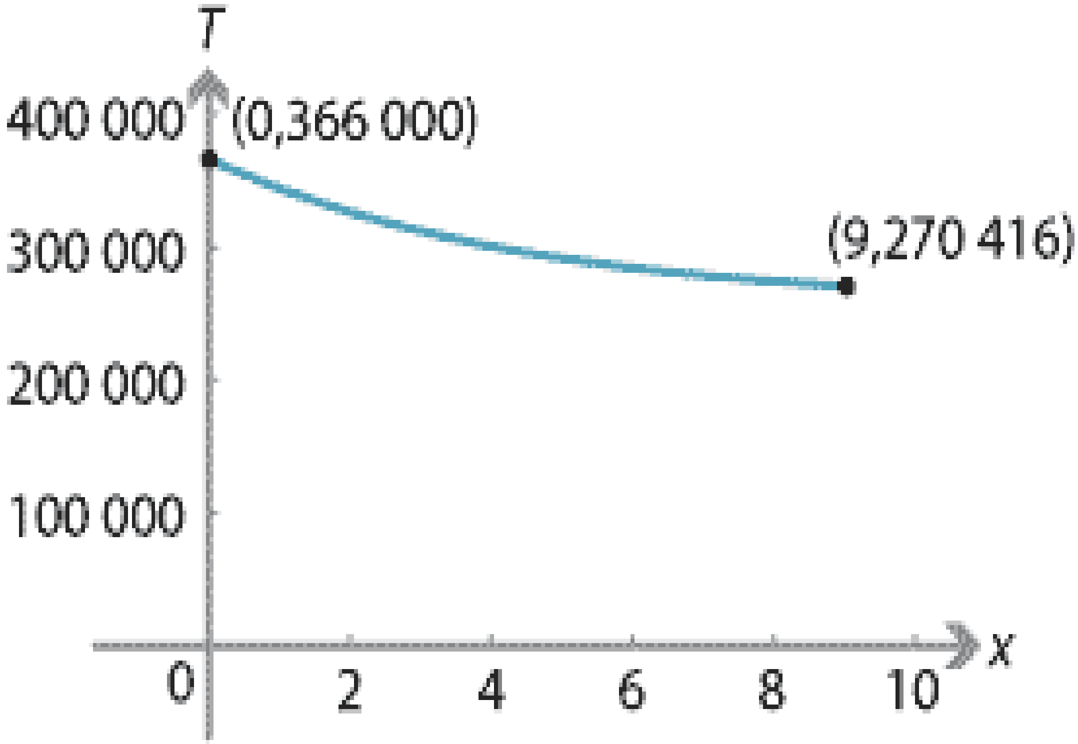
T=24000(9−x)+2500036+x2−−−−−−√,for 0≤x≤9.T=24000(9−x)+2500036+x2,for 0≤x≤9.

We note that

T(0)=366000,T(9)=7500013−−√≈270416.T(0)=366000,T(9)=7500013≈270416.

By equation (2), the local minimum occurs at x=1447x=1447, which is outside the required domain. In fact, we have dTdx<0dTdx<0, for all x∈[0,9]x∈[0,9]. The minimum cost is

T(9)=7500013−−√≈$270416.T(9)=7500013≈$270416.



Note. In parts 1 and 2, the minimum occurs at a local minimum. But, in part 3, the minimum occurs at an endpoint.

The following example has reasonably demanding algebra and involves some geometry, but the result is surprisingly neat.

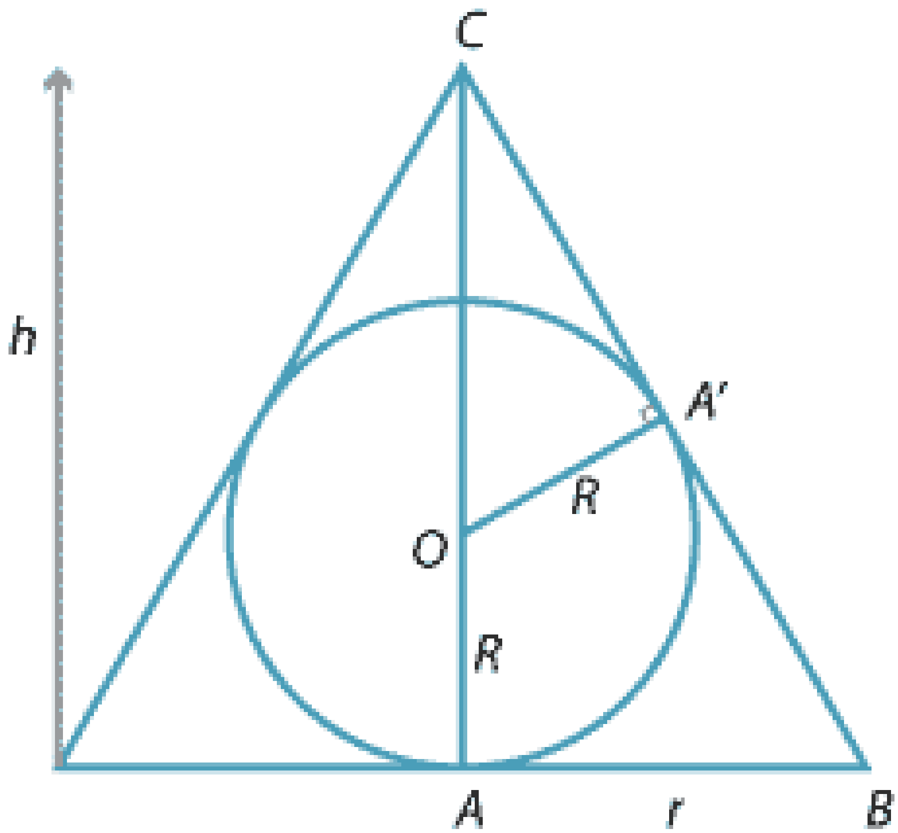
Example

A right cone is circumscribed around a given sphere. Find when its volume is a minimum.

Solution

Step 1.

The following diagram shows a vertical cross-section of the cone and sphere.



The sphere has radius RR, which we treat as a constant. The cone has radius rr and height hh. These are variables. From the geometry, we must have h>2R>0h>2R>0 and r>R>0r>R>0. The centre of the sphere is marked by OO. The radius OA′OA′ is drawn perpendicular to CBCB.

Step 2.

We will find hh in terms of rr and RR. We begin by noting that OC=h−ROC=h−R. By using Pythagoras' theorem in △OCA′,△OCA′, we get CA′=h2−2hR−−−−−−−−√CA′=h2−2hR. Since △CA′O△CA′O is similar to △CAB△CAB (AAA), we can write

CA′CA=OA′BA.CA′CA=OA′BA.

Hence,

h2−2hR−−−−−−−−√h=Rr.h2−2hRh=Rr.

Solving for hh, we obtain

h2−2hRh2r2(h2−2hR)r2h2−2hRr2r2h−2Rr2h(r2−R2)h=R2r2=h2R2=h2R2=hR2=2r2R=2r2Rr2−R2.(square both sides)(cross-multiply)(as h≠0)h2−2hRh2=R2r2(square both sides)r2(h2−2hR)=h2R2(cross-multiply)r2h2−2hRr2=h2R2r2h−2Rr2=hR2(as h≠0)h(r2−R2)=2r2Rh=2r2Rr2−R2.

The volume of the cone is given by V=13πr2hV=13πr2h. Substituting for hh, we obtain

V=2πr4R3(r2−R2).V=2πr4R3(r2−R2).

We have now expressed the volume in terms of the one variable rr.

Step 3.

We have

dVdr=4πRr3(r2−2R2)3(r2−R2)2.dVdr=4πRr3(r2−2R2)3(r2−R2)2.
So dVdr=0dVdr=0 implies that r3(r2−2R2)=0r3(r2−2R2)=0, which implies that r=0r=0 or r=2–√Rr=2R. Clearly, r=2–√Rr=2R is the solution we want.

Step 4.

Using

dVdr=4πRr3(r2−2R2)3(r2−R2)2,dVdr=4πRr3(r2−2R2)3(r2−R2)2,
we can complete the following gradient diagram.

Value of rr		2–√R2R	
Sign of dVdrdVdr	-	0	+
Slope of graph	\ \	—	/ /

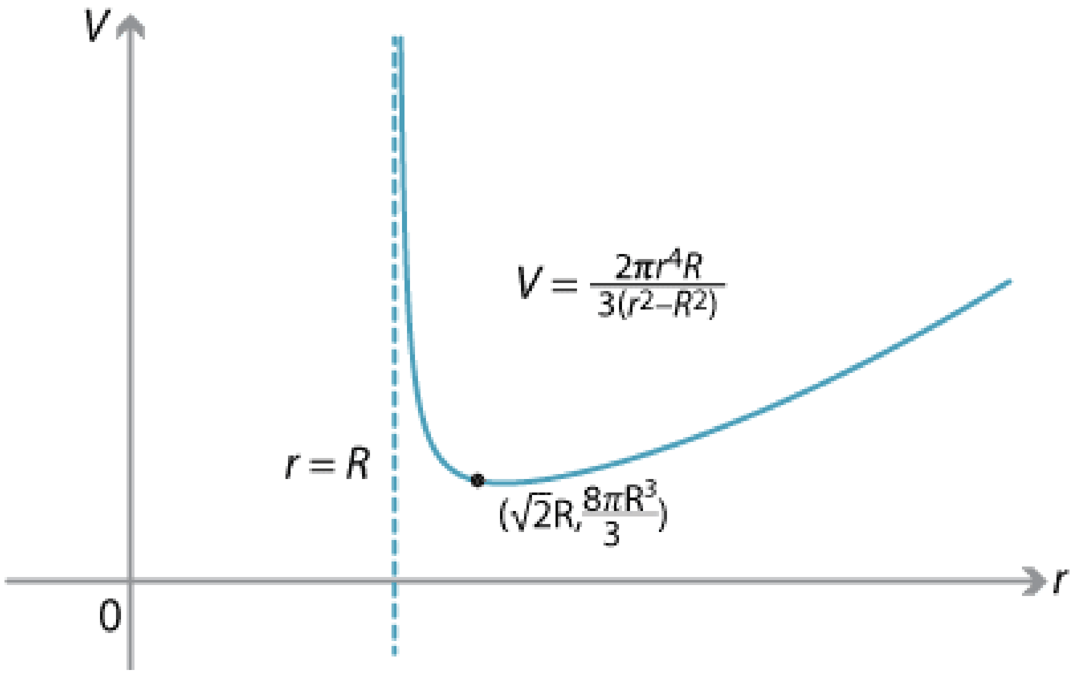
Alternatively, we can use the second derivative test. We have

d2Vdr2=4πR(r6−3r4R2+6r2R4)3(r2−R2)3.d2Vdr2=4πR(r6−3r4R2+6r2R4)3(r2−R2)3.

Substituting r=2–√Rr=2R gives

d2Vdr2=32πR3>0.d2Vdr2=32πR3>0.

Hence, we have a local minimum at r=2–√Rr=2R.
The graph of VV against rr is as follows. There is a vertical asymptote at r=Rr=R, and the graph approaches a parabola with
equation V=2πR3r2V=2πR3r2 as rr becomes very large.



Exercise 10

Find the maximum area of a rectangle that can be inscribed in the ellipse x216+y29=1x216+y29=1. Assume that the sides of the rectangle are parallel to the axes.

Find the maximum area of a rectangle that can be inscribed in the ellipse x2a2+y2b2=1x2a2+y2b2=1. Assume that the sides of the rectangle are parallel to the axes.

Exercise 11

A hollow cone has base radius RR and height HH. What is the volume of the largest cylinder that can be placed under it?